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Unsteady MHD Viscous Incompressible Bingham Fluid Flow with Hall Current

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Abstract

An electrically conducting viscous incompressible Bingham fluid bounded by two parallel non-conducting plates has been investigated in the presence of Hall current. The fluid motion is uniform at the upper plate and the uniform magnetic field is applied perpendicular to the plate. The lower plate is fixed while upper plate moves with a constant velocity. The governing equations have been non-dimensionalzed by using usual transformations. The obtained governing non-linear coupled partial differential equations have been solved by using explicit finite difference technique. The numerical solutions are obtained for momentum and energy equations. The influence of various interesting parameters on the flow has been analyzed and discussed through graph in details. The values of Local Nusselt number, Average Nusselt number, local Skin- Friction, Average Skin-Friction for different physical parameters are also illustrated in the form of graph.

Key words

MHD Flow, Hall parameter, Explicit Finite difference technique, Stability analysis.

Nomenclature

u, w	Velocity components
T_1, T_2	Temperature at lower and upper plates

B_o	Uniform magnetic field
U, W	Primary and secondary velocity
θ	Dimensionless temperature
τ_D	Dimensionless Bingham number
τ	Dimensionless time
m	Hall parameter
H_a	Hartmann number
P_r	Prandtl number
E_c	Eckert number

1. Introduction

The study about magneto hydrodynamic (MHD) flow has received considerable attention to many researchers due to its many industrial applications such as the use of MHD pumps in chemical industry technology for filtration and purification process, the operations of MHD accelerators, aerodynamics heating, electrostatics precipitation, polymer technology, petroleum industry and fluid droplets sprays. The steady MHD flow between two infinite parallel stationary plates in the presence of a transverse uniform magnetic field was first studied Attia (2007).

Such type of flow can be used in Civil engineering point of view. For bridge construction, the water flow between two piers can be measured and also the appropriate distance between the two piers can be measured.

Last few decades, great emphasis had been laid on continuum mechanics with paying particular attention on polymer solutions and polymer melts. But many geological and industry materials such as mud, lava, painting oil, drilling mud, cement, sludge, grease, granular suspensions, chocolate and paper pulp, which are frequently used in industrial problems which includes viscoplastic materials such as Bingham plastic named Bingham.

It is of special class, for which the shear stress beyond the yield stress is linearly proportional to the shear rate. If the yield stress tends to zero, the Bingham plastic fluid can entirely be treated as Newtonian fluid. But viscoplastic Couette flow, more precisely Bingham-Couette flow under the action of magnetic field applied perpendicularly has application in MHD power generators, MHD pumps, accelerators, electrostatic precipitation. In the study of the channel flow of the Bingham fluid, Friggard (1994) mentioned that an infinitesimal perturbation to the flow should displace the yield surfaces but otherwise leave them intact, since the unyielded region is "an elastic solid that would not break up". Attia (2004) studied the effects of Hall current on unsteady MHD Couette flow and heat transfer on Bingham fluid with suction and injection. Sahoo *et al.* (2010) studied the heat and mass transfer in MHD flow of a viscous fluid past a vertical plane under oscillatory suction velocity with heat source. Sahoo *et al.* (2011) investigated three dimensional MHD free convective flow with heat and mass transfer through a porous medium with periodic permeability. Also *Sahoo et al.(2011)* studied unsteady two dimensional MHD flow and heat transfer of an elastic-viscous liquid medium with source/sink. Panda *et al.*(2012) examined the heat and mass transfer on MHD flow through porous media over an accelerated surface in the presence of suction and blowing. In the following year Sahoo *et al.* (2013) showed the MHD fixed convection stagnation point flow and heat transfer in a porous medium. Naik *et al.* (2014) studied the effect of Hall current on unsteady MHD free connective Couette flow of Bingham fluid with thermal radiation. Crank Nicolson finite difference technique was used to obtain exact solution for velocity and temperature field with the effect of thermal radiation and Hall parameter.

In this paper, our aim is to study the finite difference solution of unsteady MHD viscous incompressible Bingham fluid flow with hall current. The system is considered as such that the upper plate is moving with a uniform velocity while the lower plate is fixed. A constant pressure gradient act on the plastic flow and uniform magnetic field is applied perpendicular to the plates. Very small value of Magnetic Reynolds Number is assumed to neglect the strong effect of induced magnetic field. The governing momentum and energy equations are solved numerically using the explicit finite difference approximations. Eventually interesting effects on velocity and temperature distributions, Skin friction and Nusselt number at both plates for Bingham fluid is observed. Such type of model can be used for fluid flow between two piers.

2. Mathematical formulation

The physical configuration and the boundary condition of the problem is shown in Fig: 1. The fluid is assumed to be laminar, incompressible and obeying Bingham model and flows between two infinite horizontal plates. These plates are located at the $y = \pm h$ planes and extend from $x = 0$ to ∞ and

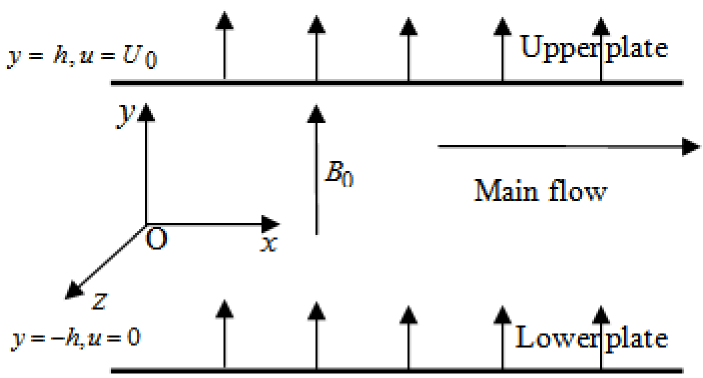


Fig: 1. The geometrical configuration

from $z=0$ to ∞. The upper plate moves with a uniform velocity U_0 while the lower plate is stationary. Both the upper and lower plates are kept at two constant temperatures are T_2 and T_1 respectively, with $T_2 > T_1$. A constant pressure gradient applied in the x-direction, and a uniform magnetic field B_0 is applied in the positive y-direction and is assumed undisturbed as the induced magnetic field is neglected by assuming a very small magnetic Reynolds number. Due to consideration of Hall Effect, a z component for the velocity is expected to arise. Thus the fluid velocity vector is $\mathbf{q} = u\hat{i} + v\hat{j} + w\hat{k}$. By using generalized Ohm's Law, the unsteady MHD Bingham fluid flows are governed by the following equations is given by;

Continuity equation: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ (1)

Momentum equation in x axis: $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) - \frac{1}{\rho}\left[\frac{\sigma B_o^2}{1+m^2}(u+mw)\right]$ (2)

Momentum equation in z axis: $\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} = \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial w}{\partial y}\right) - \frac{1}{\rho}\left[\frac{\sigma B_o^2}{1+m^2}(w-mu)\right]$ (3)

Energy equation: $\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho c_p}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] + \frac{1}{\rho c_p}\left[\frac{\sigma B_o^2}{1+m^2}(u^2+w^2)\right]$ (4)

$\mu = \mathrm{K} + \frac{\tau_o}{\sqrt{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2}}$ is the apparent viscosity of Bingham fluid

with the corresponding boundary conditions are

$$
t > 0, \quad
\begin{array}{ll}
u = 0,\ w = 0, \mathrm{T} = \mathrm{T}_1 & \text{at } \mathrm{x} = 0 \\
u = 0,\ w = 0, \mathrm{T} = \mathrm{T}_1 & \text{at } y = -h \\
u = U_o, w = 0, \mathrm{T} = \mathrm{T}_2 & \text{at } y = h
\end{array}
$$

To obtain the governing equations and the boundary condition in dimensionless form the following non-dimensional quantities are used as;

$$X = \frac{x}{h}, Y = \frac{y}{h}, Z = \frac{z}{h}, U = \frac{u}{U_o}, V = \frac{v}{U_o}, W = \frac{w}{U_o}, P = \frac{p}{\rho U_o^2}, \tau = \frac{tU_o}{h}, \theta = \frac{T - T_1}{T_2 - T_1}, \bar{\mu} = \frac{\mu}{k}$$

Using the above non-dimensional variables in equations (1- 4) and boundary conditions it can be written as (where hat is dropped)

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \qquad (5)$$

$$\frac{\partial U}{\partial \tau}+U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}=-\frac{dP}{dX}+\frac{1}{R_{\mathrm{E}}}\left[\frac{\partial}{\partial Y}\left(\mu\frac{\partial U}{\partial Y}\right)-\frac{{H_a}^2}{\left(1+m^2\right)}\left(U+mW\right)\right] \tag{6}$$

$$\frac{\partial W}{\partial \tau}+U\frac{\partial W}{\partial X}+V\frac{\partial W}{\partial Y}=\frac{1}{R_{\mathrm{E}}}\left[\frac{\partial}{\partial Y}\left(\mu\frac{\partial W}{\partial Y}\right)-\frac{{H_a}^2}{\left(1+m^2\right)}\left(W-mU\right)\right] \tag{7}$$

$$\frac{\partial \theta}{\partial \tau}+U\frac{\partial \theta}{\partial X}+V\frac{\partial \theta}{\partial Y}=\frac{1}{P_r}\frac{\partial^2\theta}{\partial Y^2}+\mathrm{E}_c\mu\left[\left(\frac{\partial U}{\partial Y}\right)^2+\left(\frac{\partial W}{\partial Y}\right)^2\right]+\frac{{H_a}^2E_c}{\left(1+m^2\right)}\left(U^2+W^2\right) \tag{8}$$

$$\mu=1+\frac{\tau_D}{\sqrt{\left(\frac{\partial U}{\partial Y}\right)^2+\left(\frac{\partial W}{\partial Y}\right)^2}}$$

And the dimensionless boundary conditions are;

$$\tau>0,\quad \begin{aligned} &U=0,\ \mathrm{W}=0,\theta=0 \quad \text{at X}=0\\ &U=0,\ \mathrm{W}=0,\theta=0 \quad \text{at Y}=-1\\ &U=1,\ \mathrm{W}=0,\theta=1 \quad \text{at Y}=1 \end{aligned}$$

3. Shear Stress and Nusselt Number

From the velocity field, the effects of various parameters on Shear Stress have been studied. Shear Stress in x direction for stationary wall is $\tau_{w1}=\left[\mu\sqrt{\left(\frac{\partial U}{\partial Y}\right)^2+\left(\frac{\partial W}{\partial Y}\right)^2}\right]_{Y=-1}$ and for moving wall is $\tau_{w2}=\left[\mu\sqrt{\left(\frac{\partial U}{\partial Y}\right)^2+\left(\frac{\partial W}{\partial Y}\right)^2}\right]_{Y=1}$ .From the temperature field, the effects of various parameters on Nusselt number have been analyzed. Nusselt Number for stationary wall is $N_{u1}=\frac{\left(2\frac{\partial T}{\partial Y}\right)_{Y=-1}}{-T_m}$ and for moving wall is $N_{u2}=\frac{\left(2\frac{\partial T}{\partial Y}\right)_{Y=1}}{-\left(T_m-1\right)}$, where T_m is the dimensionless mean fluid temperature and is given by $T_m=\left(2\int_{-1}^{1}U\theta dY\Big/\int_{-1}^{1}UdY\right)$

4. Numerical Solutions

To solve the non-dimensional system by explicit finite difference method, a set of finite difference equations is required. For this reason the area within the boundary layer is divided into a grid or mesh of lines parallel to x and y axis. Where x axis is taken along the plate and y axis is normal to the plate as shown in Fig: 2. It is considered that the plate of height $X_{\max}(=40)$ i.e. X varies from 1 to 40 and regard $Y_{\max}(=2)$ i.e. Y varies from 1 to 2.

The number of grid spacing in both directions are $m=40, n=40$. Hence the constant mesh size along x and y directions are $\Delta x = 1.0$ and $\Delta y = 0.05$ respectively with smaller time step $\Delta\tau = 0.0001$.

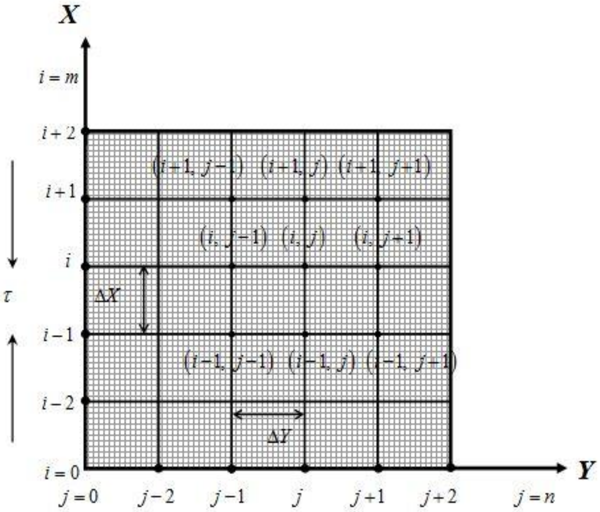


Fig: 2. Finite difference system grid

Let U', W' and θ' denotes the value of U, W and θ and the end of the time-step respectively. Using explicit finite difference, the following appropriate set of finite difference equation are obtained as;

$$\frac{U_{i,j}-U_{i-1,j}}{\Delta x}+\frac{V_{i,j}-V_{i-1,j}}{\Delta y}=0 \tag{9}$$

$$\frac{U_{i,j}{}'-U_{i,j}}{\Delta\tau}+U_{i,j}\frac{U_{i,j}-U_{i-1,j}}{\Delta X}+V_{i,j}\frac{U_{i,j+1}-U_{i,j}}{\Delta Y}= -\frac{dP}{dX}+\frac{1}{R_{\mathrm{E}}}\left[\left(\frac{\mu_{i,j+1}-\mu_{i,j}}{\Delta Y}\right)\left(\frac{U_{i,j+1}-U_{i,j}}{\Delta Y}\right)+\mu_{i,j}\left(\frac{U_{i,j+1}-2U_{i,j}+U_{i,j-1}}{\Delta Y^2}\right)-\frac{H_a{}^2}{1+m^2}\left(U_{i,j}+mW_{i,j}\right)\right] \tag{10}$$

$$\frac{W_{i,j}{}'-W_{i,j}}{\Delta\tau}+U_{i,j}\frac{W_{i,j}-W_{i-1,j}}{\Delta X}+V_{i,j}\frac{W_{i,j+1}-W_{i,j}}{\Delta Y}= \frac{1}{R_{\mathrm{E}}}\left[\left(\frac{\mu_{i,j+1}-\mu_{i,j}}{\Delta Y}\right)\left(\frac{W_{i,j+1}-W_{i,j}}{\Delta Y}\right)+\mu_{i,j}\left(\frac{W_{i,j+1}-2W_{i,j}+W_{i,j-1}}{\Delta Y^2}\right)-\frac{H_a{}^2}{1+m^2}\left(W_{i,j}-mU_{i,j}\right)\right] \tag{11}$$

$$\frac{\theta_{i,j}{}'-\theta_{i,j}}{\Delta\tau}+U_{i,j}\frac{\theta_{i,j}-\theta_{i-1,j}}{\Delta X}+V_{i,j}\frac{\theta_{i,j+1}-\theta_{i,j}}{\Delta Y}=\frac{1}{P_r}\left[\frac{\theta_{i,j+1}-2\theta_{i,j}+\theta_{i,j-1}}{\Delta Y^2}\right] +E_c\left(\mu_{i,j}\right)\left[\left(\frac{U_{i,j+1}-U_{i,j}}{\Delta Y}\right)^2+\left(\frac{W_{i,j+1}-W_{i,j}}{\Delta Y}\right)^2\right]+\frac{H_a{}^2E_c}{1+m^2}\left[\left(U_{i,j}\right)^2+\left(W_{i,j}\right)^2\right] \tag{12}$$

With the finite difference boundary conditions

$$U_{i,L}=0, W_{i,L}=0, \theta_{i,L}=0 \ \text{ at } L=-1$$
$$U_{i,L}=1, W_{i,L}=0, \theta_{i,L}=1 \ \text{ at } L=1$$

The numerical values of local shear stress, local Nusselt number are evaluated by **Five point** approximate formula and the average shear stress, average Nusselt number are calculated by **Simpson's** $\frac{1}{3}$ integration rule. The stability conditions of the method are; $\frac{U\Delta\tau}{\Delta X}+\frac{|V|\Delta\tau}{\Delta Y}+\frac{\Delta\tau}{2R_E}\frac{H_a^{\,2}}{(1+m^2)}\le 1$, $\frac{U\Delta\tau}{\Delta X}+\frac{|V|\Delta\tau}{\Delta Y}+\frac{2\Delta\tau}{P_r(\Delta Y)^2}\le 1$ and the convergence criteria of the problem are $H_a\le 10, R_E\ge 0.00251, m\ge 1, P_r\ge 0.08$ with $E_c=0.1$.(details are not shown for brevity).

5. Results and Discussion

The obtained governing equations are non-linear, coupled partial differential equations which cannot be solved analytically. That's why, explicit finite difference technique has been used to solve these equations. To obtain the numerical solutions, the computations have been carried out up to $\tau=0.1$ to $\tau=20.00$. The results of computation show little changes for $\tau=0.1$ to $\tau=4.50$,but after that until $\tau=20.00$ the results remain approximately same. In order to get the clear concept of physical properties of the problem, the effects of two parameters namely Hall parameter (m), Hartmann number (H_a) in the presence of Reynolds number (R_E), Prandtl number (P_r) and Eckert number (E_c) are represented graphically through Figs: (3- 6). The effects of Hall current (m) on Shear Stress both at stationary and moving plate are presented in Fig.3(a-b). It is observed that Shear Stress at both plates increase with the increase of m .This is due to the fact an increase in m decreases effective conductivity $(=\sigma/(1+m^2))$, hence magnetic dumping force on U . The effects of Hall parameter (m) on Nusselt number both at stationary and moving plate are elucidated in Fig. 4(a-b). As U and W increases with the increase of m , joule and viscous dissipations also increases for which temperature increases. But the reverse effects is observed for Nusselt number. The effects of Hartmann number (H_a) on shear stress for both plates are shown in Fig: 5(a-b).

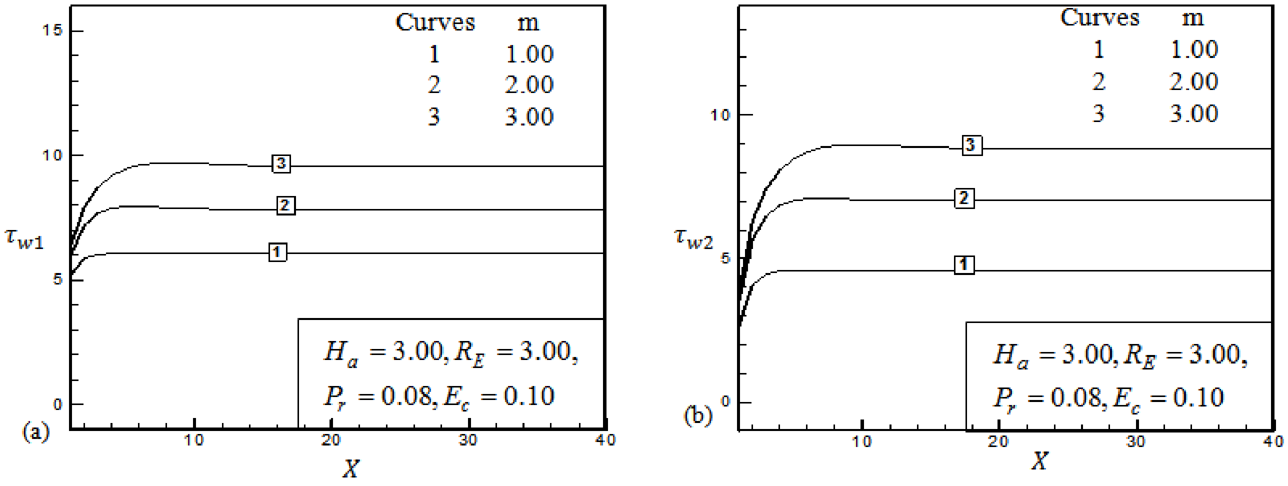
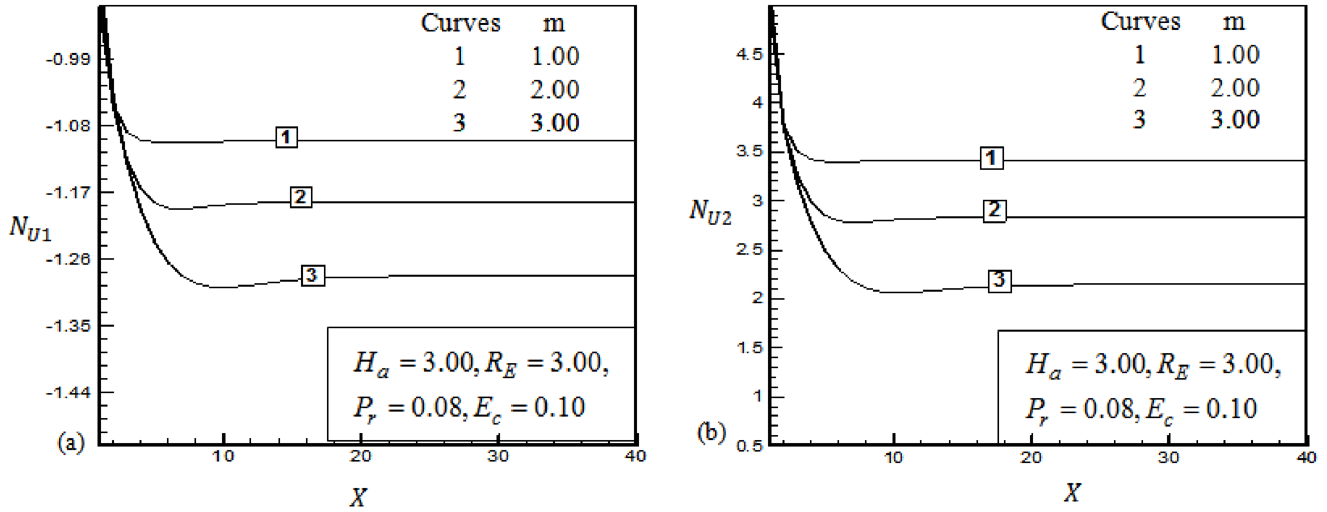


Fig: 3 Illustration of Shear Stress at (a) Stationary wall (b) Moving wall for different values of Hall Parameter



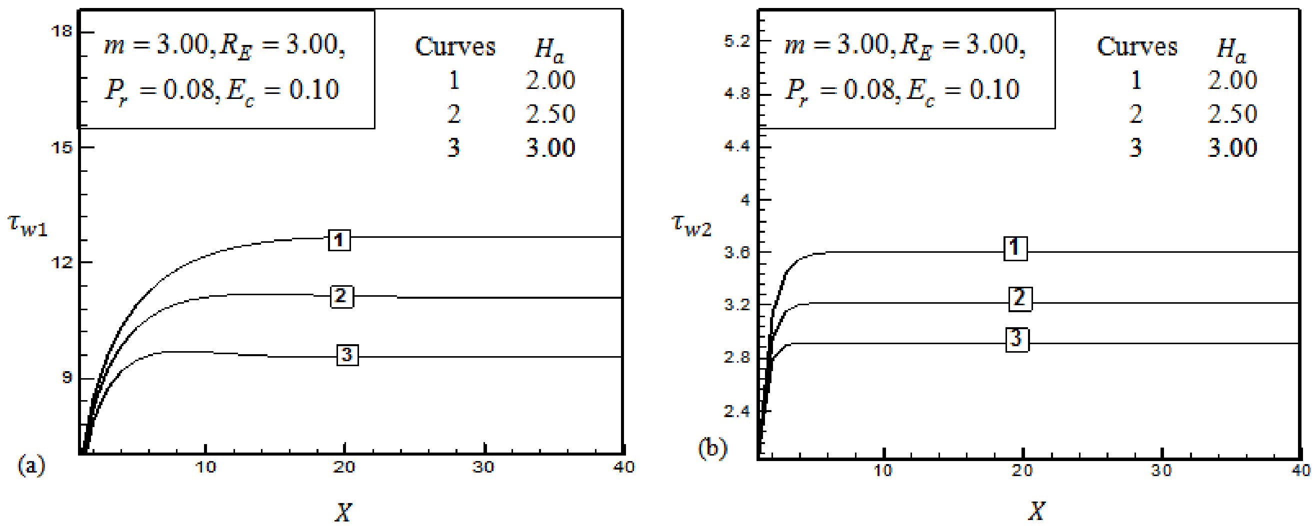


Fig: 4 Illustration of Nusselt Number at (a) Stationary wall (b) Moving wall for different values of Hall Parameter

Fig: 5 Illustration of Shear Stress at (a) Stationary wall (b) Moving wall for different values of Hartmann number

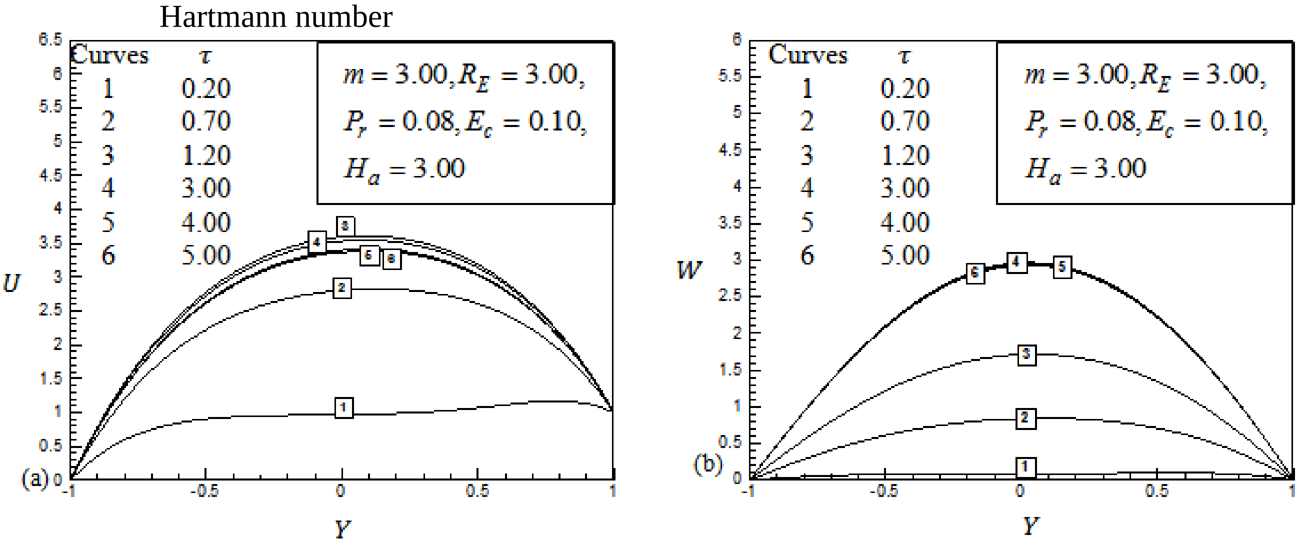


Fig: 6 Illustration of Nusselt Number at (a) Stationary wall (b) Moving wall for different values of Hartmann number

It is seen that with the increase of Hartmann number (H_a) shear stress decreases, showing the effect of dragging the magnetic field as Hartmann number gives a measure of the relative importance of drag forces resulting from magnetic induction and viscous forces. The effects of Hartmann number on Nusselt number are shown in Fig: 6(a-c). It is observed that with increase of (H_a) Nusselt number at both plates increases. Due to the incitement of convection by the magnetic field, results a gradual increase of Nusselt number.

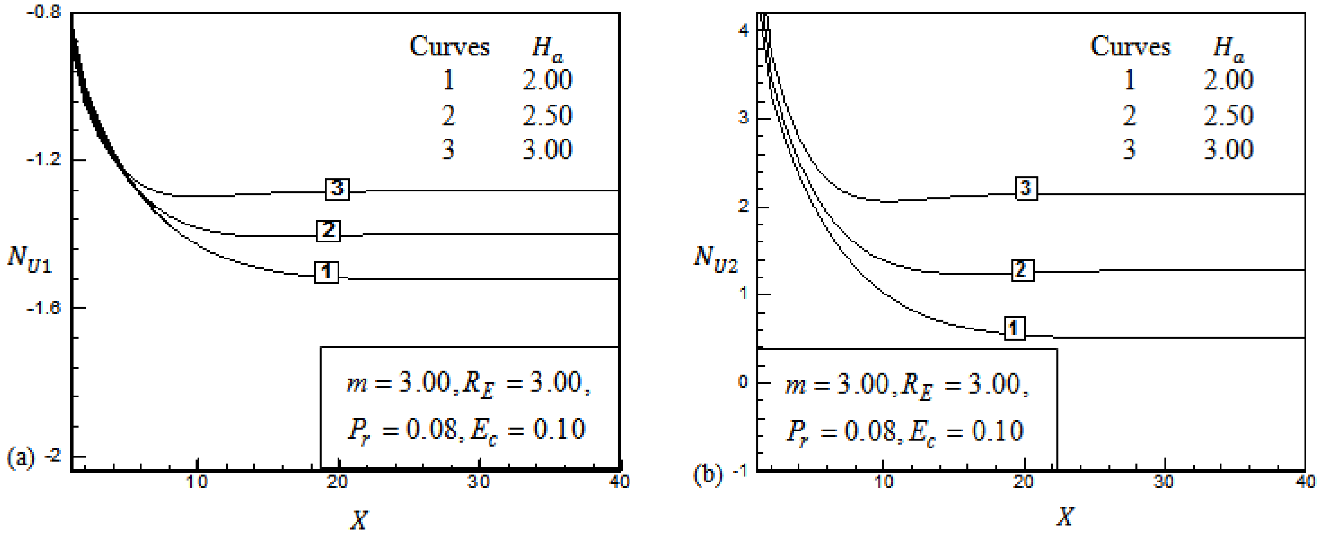


Fig: 7 Illustration of Time Variation for (a) Primary Velocity (b) Secondary Velocity

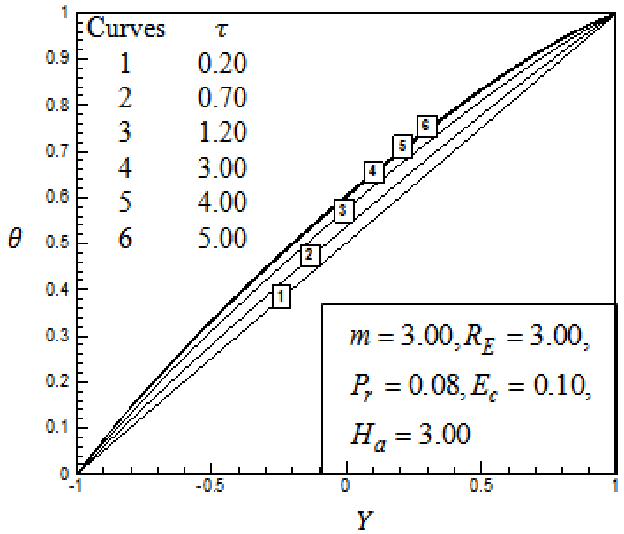


Fig: 8 Illustration of Time Variation for temperature

The profile of primary velocity, secondary velocity and temperature for different times is shown in Fig: 7-8. It is seen from Fig: 7(a) that primary velocity does not reach its steady state monotonically. It increases with time up till a maximum value and then decrease up to steady state. But from Fig: 7(b) and Fig: 8 it is clear that both secondary velocity and temperature profiles reach their steady state monotonically. It also should be mentioned that primary velocity reaches the steady state faster than secondary velocity which, in turn, reaches steady state faster than temperature.

Conclusion

In this research, the explicit finite difference solution of unsteady MHD viscous incompressible Bingham fluid flow bounded by two electrically non-conducting plate in the presence of Hall current, Hartmann number, Reynolds number, Eckert number and Prandtl number has been investigated. For brevity, the effect of Eckert number and Prandtl number are not shown. The physical properties are illustrated graphically for different values of corresponding parameters. Among them some important findings of this investigation are mentioned here;

1. The Shear stress at stationary wall and moving wall increases with the increase of m.
2. The Nusselt number at stationary wall and moving wall decreases with the increase of m.
3. The Shear stress at stationary wall and moving wall decreases with the increase of H_a.
4. The Nusselt number at stationary wall and moving wall increases with the increase of H_a.
5. Primary velocity reaches steady state briskly in the comparison with secondary velocity and temperature.

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