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A New Recursive Approach for The Analysis and Optimal Control of Linear Time Varying Dynamic Systems via Taylor Series

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Abstract

The paper presents a new recursive technique using Taylor series in state analysis and optimal control problems of time varying systems. The operational matrix for integration of fixed interval lengths and Kronecker product approach is required in traditional methods, whereas this method does not follow the same strategy. Numerical examples are treated to justify the proposed method and the results obtained are compared with the exact solutions via related tables and graphs. However, the recursive technique is proved to be acceptable over other methods because of its elegance and simplicity and thereby makes Taylor series as a powerful tool.

Key words

Taylor series, recursive algorithm, state space, time varying system, homogeneous system, optimal gain, quadratic performance index, approximation error.

1. Introduction

The Taylor expansion was introduced a long days back in 1715 by Brook Taylor to approximate a function as an infinite sum of terms calculated from the values of its different order derivatives at a single point. Despite the concept being age old, over three centuries it has proved its potential in many areas of mathematics. Here, the state analysis and optimal control problem has been investigated via the Taylor series in recursive way. Although it had been employed in different way involving operational matrices in earlier papers, [1-4], but such matrix and subsequently matrix inversions are entirely avoided in proposed algorithm.

State space problem of time varying systems has already been dealt with orthogonal functions such as Haar, Walsh, block pulse, and as well as by Legendre polynomial, Taylor series even with triangular functions, hybrid functions. The Taylor series were used for delay systems, singular systems, linear time invariant and nonlinear systems [2-6, 8]. Optimal Control problems are studied using several polynomials such as Legendre, Laguerre, Taylor, Chebyshev [2, 7] etc. However, these works never focused on recursion so that the computation can be much faster.

2. Function approximation via Taylor series

Any function $f(t)$ defined over $t \in [0, T)$ can be approximated by a Taylor series around a specific point where it is differentiable μ_k, $\mu_k \in [0, T)$ by,

$$f(t) \approx f(\mu_k) + \dot{f}(\mu_k)(t - \mu_k) + \ddot{f}(\mu_k)\frac{(t-\mu_k)^2}{2!} + \dddot{f}(\mu_k)\frac{(t-\mu_k)^3}{3!} + \ldots \triangleq \bar{f}(t) \quad \text{(say)} \tag{1}$$

Let $t \in [0, T)$ is divided into m sub-intervals of equal width h, so, $h = \frac{T}{m}$. Let, $\mu_k = kh$, where k = 0, 1, 2,… , (m-1), equation (1) can be modified for first and second order Taylor series into,

$$f(t) \approx f(kh) + \dot{f}(kh)(t - kh) \triangleq \bar{f}1(t) \quad \text{(say)} \tag{2}$$

and $$f(t) \approx f(kh) + \dot{f}(kh)(t - kh) + \ddot{f}(kh)\frac{(t-kh)^2}{2!} \triangleq \bar{f}2(t) \quad \text{(say)} \tag{3}$$

Further, (2) and (3) can be restructured by, $$\bar{f}1(t) = \bar{f}1(kh) + \dot{\bar{f}}1(kh)(t - kh) \tag{4}$$

and $$\bar{f}2(t) = \bar{f}2(kh) + \dot{\bar{f}}2(kh)(t - kh) + \ddot{\bar{f}}2(kh)\frac{(t-kh)^2}{2!} \tag{5}$$

assuming that approximation $f(t) \triangleq \bar{f}1(t)$ and $f(t) \triangleq \bar{f}2(t)$ i. e., $f(kh) \triangleq \bar{f}1(kh)$ and $f(kh) \triangleq \bar{f}2(kh)$ using first and second order Taylor series for $t=kh$. From (4) and (5), the approximation can be determined by putting $t=(k+1)h$, as $$\bar{f}1\{(k+1)h\} = \bar{f}1(kh) + h\,\dot{\bar{f}}1(kh) \tag{6}$$

and $$\bar{f}2\{(k+1)h\} = \bar{f}2(kh) + h\,\dot{\bar{f}}2(kh) + \frac{h^2}{2!}\ddot{\bar{f}}2(kh) \tag{7}$$

Now, assuming $\mu_k = (k+1)h$, equation (4) and (5) becomes

$$\bar{f}1(t) = \bar{f}1\{(k+1)h\} + \dot{\bar{f}}1\{(k+1)h\}\left[t - \{(k+1)h\}\right] \tag{8}$$

$$\bar{f}2(t) = \bar{f}2\{(k+1)h\} + \dot{\bar{f}}2\{(k+1)h\}\left[t - \{(k+1)h\}\right] + \ddot{\bar{f}}2\{(k+1)h\}\frac{\left[t - \{(k+1)h\}\right]^2}{2!} \tag{9}$$

The approximation in (k+1)-th interval $t \in (kh, (k+1)h)$ can be determined for sample at $t=kh$ as,

$$\bar{f}1(kh) = \bar{f}1\{(k+1)h\} - h\,\dot{\bar{f}}1\{(k+1)h\} \tag{10}$$

$$\bar{f}2(kh) = \bar{f}2\{(k+1)h\} - h\,\dot{\bar{f}}2\{(k+1)h\} + \frac{h^2}{2!}\ddot{\bar{f}}2\{(k+1)h\} \tag{11}$$

This can give an idea of approximation for different index *k*, *k*=(*m*-1), (*m*-2), …, 2, 1, 0. Thus, equations (6), (10) and (7), (11) results the approximation using first and second order Taylor series respectively along time axis but only the difference is (6), (7) is determined for *k*=0, 1, …, (*m*-1) while (10) and (11) is determined for *k*=(*m*-1), (*m*-2), …, 2, 1, 0.

Now, state equations of linear time varying systems are solved using both first order and second order Taylor series based upon the recursive technique elaborated in equations (6) and (7) respectively.

3. Analysis of linear time varying system

The state equation of a linear non-homogenous time varying (LTI) system is

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\,\mathbf{x}(t) + \mathbf{B}(t)\,\mathbf{u}(t) \quad \text{and} \quad \mathbf{x}(0) = \mathbf{x_0} \tag{12}$$

Differentiating (22), $\ddot{\mathbf{x}}(t) = \dot{\mathbf{A}}(t)\,\mathbf{x}(t) + \mathbf{A}(t)\,\dot{\mathbf{x}}(t) + \dot{\mathbf{B}}(t)\,\mathbf{u}(t) + \mathbf{B}(t)\,\dot{\mathbf{u}}(t)$ (13)

The state approximation in first and second order Taylor domain being $\mathbf{x}(t) \triangleq \overline{\mathbf{x}}1(t)$ and $\mathbf{x}(t) \triangleq \overline{\mathbf{x}}2(t)$, i. e., at $t = kh$, $\mathbf{x}(kh) \triangleq \overline{\mathbf{x}}1(kh)$ and $\mathbf{x}(kh) \triangleq \overline{\mathbf{x}}2(kh)$, (12), (13) are modified into

$$\dot{\overline{\mathbf{x}}}1(kh) = \mathbf{A}(kh)\,\overline{\mathbf{x}}1(kh) + \mathbf{B}(kh)\,\mathbf{u}(kh) \tag{14}$$

and $$\dot{\overline{\mathbf{x}}}2(kh) = \mathbf{A}(kh)\,\overline{\mathbf{x}}2(kh) + \mathbf{B}(kh)\,\mathbf{u}(kh) \tag{15}$$

$$\ddot{\overline{\mathbf{x}}}2(kh) = \dot{\mathbf{A}}(kh)\,\overline{\mathbf{x}}2(kh) + \mathbf{A}(kh)\,\dot{\overline{\mathbf{x}}}2(kh) + \dot{\mathbf{B}}(kh)\,\mathbf{u}(kh) + \mathbf{B}(kh)\,\dot{\mathbf{u}}(kh) \tag{16}$$

Following (6) and (7), the recursive equations for states via first and second order Taylor series is

$$\overline{\mathbf{x}}1\{(k+1)\,h\} = \overline{\mathbf{x}}1(kh) + h\,\dot{\overline{\mathbf{x}}}1(kh) \tag{17}$$

and $$\overline{\mathbf{x}}2\{(k+1)\,h\} = \overline{\mathbf{x}}2(kh) + h\,\dot{\overline{\mathbf{x}}}2(kh) + \frac{h^2}{2!}\,\ddot{\overline{\mathbf{x}}}2(kh) \tag{18}$$

Substituting (14) in equations (17) and in similar way (15), (16) in (18),

$$\overline{\mathbf{x}}1\{(k+1)\,h\} = \left[\mathbf{I} + h\mathbf{A}(kh)\right]\overline{\mathbf{x}}1(kh) + h\mathbf{B}(kh)\,\mathbf{u}(kh) \tag{19}$$

$$\begin{aligned}\overline{\mathbf{x}}2\{(k+1)\,h\} &= \left(\mathbf{I} + h\mathbf{A}(kh) + \frac{h^2}{2!}\dot{\mathbf{A}}(kh) + \frac{h^2}{2!}\mathbf{A}^2(kh)\right)\overline{\mathbf{x}}2(kh) \\ &+ \mathbf{u}(kh)\left(h\,\mathbf{B}(kh) + \frac{h^2}{2!}\mathbf{A}(kh)\mathbf{B}(kh) + \frac{h^2}{2!}\dot{\mathbf{B}}(kh)\right) + \frac{h^2}{2!}\mathbf{B}(kh)\,\dot{\mathbf{u}}(kh)\end{aligned} \tag{20}$$

Equations (19) and (20) provide a recursive solution [8] of the state vector $\mathbf{x}(t)$ for the linear time varying system described by (12). Knowing initial values of the states $\mathbf{x}(0) = \mathbf{x_0}$, the recursive process can be started from k = 0 and continues for any k=0, 1, …, (m-1). During computation one can choose any length of h, without fixing its value by $h = \frac{T}{m}$.

The approximation error for both the cases can be defined in mean integral square error (MISE) with respect to exact state $\mathbf{x}(t)$ using the following equations

$$\text{MISE} \triangleq \frac{1}{T}\int_0^T \left[\mathbf{x}(t) - \bar{\mathbf{x}}\mathbf{1}(t)\right]^2 \mathrm{d}t \quad \text{and} \quad \text{MISE} \triangleq \frac{1}{T}\int_0^T \left[\mathbf{x}(t) - \bar{\mathbf{x}}\mathbf{2}(t)\right]^2 \mathrm{d}t \qquad (21)$$

The method above mentioned can be implemented for online process by programming equations (19) or (20) using microprocessor or microcontroller [8].

3. Optimal control problem in Riccatti approach

For the optimal control problem let the quadratic performance index of the time varying system (12) be

$$J = \frac{1}{2}\int_0^{t_f}\left[\mathbf{x}^T(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}(t)\mathbf{u}(t)\right]dt \qquad (22)$$

where, **Q** is an nxn matrix, **R** is rxr matrix. The optimal feedback control law is given by,

$$\mathbf{u}^*(t) = -\mathbf{k}(t)\mathbf{x}(t) \qquad (23)$$

where, $\mathbf{k}(t) = \mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{P}(t)$ (24)

subjected to boundary condition $\mathbf{x}(t=0) = \mathbf{x_0}$ and $\mathbf{P}(t = t_f) = 0$ (25)

The matrix Riccati equation is

$$\dot{\mathbf{P}}(t) = \begin{bmatrix}\mathbf{P}(t) & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{M}(t)\end{bmatrix}\begin{bmatrix}\mathbf{P}(t)\\ \mathbf{I}\end{bmatrix}; \quad \mathbf{M}(t) = \begin{bmatrix}\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t) & -\mathbf{A}(t)\\ -\mathbf{A}^T(t) & -\mathbf{Q}(t)\end{bmatrix} \qquad (26)$$

Differentiating (26),

$$\ddot{\mathbf{P}}(t) = \begin{bmatrix}\dot{\mathbf{P}}(t) & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{M}(t)\end{bmatrix}\begin{bmatrix}\mathbf{P}(t)\\ \mathbf{I}\end{bmatrix} + \begin{bmatrix}\mathbf{P}(t) & \mathbf{I}\end{bmatrix}\begin{bmatrix}\dot{\mathbf{M}}(t)\end{bmatrix}\begin{bmatrix}\mathbf{P}(t)\\ \mathbf{I}\end{bmatrix} + \begin{bmatrix}\mathbf{P}(t) & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{M}(t)\end{bmatrix}\begin{bmatrix}\dot{\mathbf{P}}(t)\\ \mathbf{0}\end{bmatrix} \qquad (27)$$

Let us assume, for approximation, $\mathbf{P} \triangleq \bar{\mathbf{P}}1$ in first order Taylor domain and $\mathbf{P} \triangleq \bar{\mathbf{P}}2$ in second order Taylor domain. Equations (26) and subsequently (26), (27) is modeled as,

$$\dot{\bar{\mathbf{P}}}1(t) = \begin{bmatrix}\bar{\mathbf{P}}1(t) & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{M}(t)\end{bmatrix}\begin{bmatrix}\bar{\mathbf{P}}1(t)\\ \mathbf{I}\end{bmatrix} \qquad (28)$$

And $\dot{\bar{\mathbf{P}}}2(t) = \begin{bmatrix}\bar{\mathbf{P}}2(t) & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{M}(t)\end{bmatrix}\begin{bmatrix}\bar{\mathbf{P}}2(t)\\ \mathbf{I}\end{bmatrix}$ (29)

$$\ddot{\bar{\mathbf{P}}}2(t) = \begin{bmatrix}\dot{\bar{\mathbf{P}}}2(t) & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{M}(t)\end{bmatrix}\begin{bmatrix}\bar{\mathbf{P}}2(t)\\ \mathbf{I}\end{bmatrix} + \begin{bmatrix}\bar{\mathbf{P}}2(t) & \mathbf{I}\end{bmatrix}\begin{bmatrix}\dot{\mathbf{M}}(t)\end{bmatrix}\begin{bmatrix}\bar{\mathbf{P}}2(t)\\ \mathbf{I}\end{bmatrix} + \begin{bmatrix}\bar{\mathbf{P}}2(t) & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{M}(t)\end{bmatrix}\begin{bmatrix}\dot{\bar{\mathbf{P}}}2(t)\\ \mathbf{0}\end{bmatrix} \qquad (30)$$

As, according to boundary condition $\mathbf{P}(t = t_f) = 0$, and approximation of **P** is required within the span $t \in (0,\ t_f)$, we opt the recursion process using first and second order Taylor series in the retrogressive manner. Here, assumption is $\mathbf{P}(t_f) \triangleq \bar{\mathbf{P}}1(t_f) = 0$ and $\mathbf{P}(t_f) \triangleq \bar{\mathbf{P}}2(t_f) = 0$. Let, $t_f = T$ and $t \in (0,\ T)$ is divided into *m* intervals of length *h*.

Hence following (10) and (11), the equations below provide the approximation of **P** in first and second order Taylor domain as $\bar{\mathbf{P}}1$ and $\bar{\mathbf{P}}2$ respectively for *k*=(*m*-1), (*m*-2), …, 2, 1, 0., as,

$$\bar{\mathbf{P}}1(kh) = \bar{\mathbf{P}}1\{(k+1)h\} - h\dot{\bar{\mathbf{P}}}1\{(k+1)h\} \tag{31}$$

$$\bar{\mathbf{P}}2(kh) = \bar{\mathbf{P}}2\{(k+1)h\} - h\dot{\bar{\mathbf{P}}}2\{(k+1)h\} + \frac{h^2}{2!}\ddot{\bar{\mathbf{P}}}2\{(k+1)h\} \tag{32}$$

Hence, using these formulae, estimation can be done starting right from $\mathbf{P}(t = t_f) = 0$ of (25) to get an approximation in $t \in (0, T)$ and thus for the last iteration it results $\mathbf{P}(t = 0) = \mathbf{P}_0$. So, putting (28) in (31) results in $\bar{\mathbf{P}}1$ and further putting (29) and (30) in (32) provides approximation $\bar{\mathbf{P}}2$.

Now, optimal gain **k**(*t*) can be structured using the equation (24) and can be compared with the exact gain. Here, the approximation of **P**, i. e., $\bar{\mathbf{P}}1$ and $\bar{\mathbf{P}}2$ results an approximation of **k**. Let it be assumed as $\mathbf{k} \triangleq \bar{\mathbf{k}}1$ and $\mathbf{k} \triangleq \bar{\mathbf{k}}2$ in first and second order Taylor domain respectively. So, (24) can be written as, $\bar{\mathbf{k}}\mathbf{1}(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\bar{\mathbf{P}}\mathbf{1}(t)$; $\bar{\mathbf{k}}\mathbf{2}(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\bar{\mathbf{P}}\mathbf{2}(t)$ (33)

3.1. Optimal input and state approximation

The optimal input **(23)** can be replaced in **equation (12)**, and thereby the above state equation changes into a homogeneous equation as,

$$\dot{\mathbf{x}}(t) = [\mathbf{A}(t) - \mathbf{B}(t)\,\mathbf{k}(t)]\mathbf{x}(t) \tag{34}$$

Putting *t*=*kh*, where *k*=0, 1, …, (*m*-1) in (34) and in its differentiation,

$$\dot{\mathbf{x}}(kh) = [\mathbf{A}(kh) - \mathbf{B}(kh)\,\mathbf{k}(kh)]\mathbf{x}(kh) \tag{35}$$

$$\ddot{\mathbf{x}}(kh) = \left[\dot{\mathbf{A}}(kh) - \mathbf{B}(kh)\,\dot{\mathbf{k}}(kh) - \dot{\mathbf{B}}\,\mathbf{k}(kh)\right]\mathbf{x}(kh) + [\mathbf{A}(kh) - \mathbf{B}(kh)\,\mathbf{k}(kh)]\dot{\mathbf{x}}(kh) \tag{36}$$

Now, let (35) in first order Taylor domain be

$$\dot{\bar{\mathbf{x}}}1(kh) = [\mathbf{A}(kh) - \mathbf{B}(kh)\,\bar{\mathbf{k}}\mathbf{1}(kh)]\bar{\mathbf{x}}1(kh) \tag{37}$$

where, $\mathbf{x}(t) \triangleq \bar{\mathbf{x}}1(t)$ and *k*=0, 2, 3, …, (*m*-1) as considered earlier and $\mathbf{k} \triangleq \bar{\mathbf{k}}1$. And (35), (36) modeled in second order Taylor domain be

$$\dot{\bar{\mathbf{x}}}2(kh) = \left[\mathbf{A}(kh) - \mathbf{B}(kh)\,\bar{\mathbf{k}}\mathbf{2}(kh)\right]\bar{\mathbf{x}}2(kh) \tag{38}$$

$$\ddot{\bar{\mathbf{x}}}2(kh) = \left[\dot{\mathbf{A}}(kh) - \mathbf{B}(kh)\,\dot{\bar{\mathbf{k}}}\mathbf{2}(kh) - \dot{\mathbf{B}}(kh)\bar{\mathbf{k}}\mathbf{2}(kh)\right]\bar{\mathbf{x}}2(kh) + \left[\mathbf{A}(kh) - \mathbf{B}(kh)\,\bar{\mathbf{k}}\mathbf{2}(kh)\right]\dot{\bar{\mathbf{x}}}2(kh) \tag{39}$$

where, $\mathbf{x}(t) \triangleq \bar{\mathbf{x}}2(t)$ and k=0, 2, 3, …, (m-1), and $\mathbf{k} \triangleq \bar{\mathbf{k}}2$. Hence, putting (37) in (17),

$$\bar{\mathbf{x}}1\{(k+1)\,h\} = \left(\mathbf{I} + h\left[\mathbf{A}(kh) - \mathbf{B}(kh)\,\bar{\mathbf{k}}\mathbf{1}(kh)\right]\right)\bar{\mathbf{x}}1(kh) \tag{40}$$

And substituting (38), (39) in (18)

$$\bar{\mathbf{x}}2\{(k+1)\,h\} = \Bigg(\mathbf{I} + h\left[\mathbf{A}(kh) - \mathbf{B}(kh)\bar{\mathbf{k}}\mathbf{2}(kh)\right] + \frac{h^2}{2!}\left[\mathbf{A}(kh) - \mathbf{B}(kh)\bar{\mathbf{k}}\mathbf{2}(kh)\right]^2$$
$$+ \frac{h^2}{2!}\left[\dot{\mathbf{A}}(kh) - \mathbf{B}(kh)\,\dot{\bar{\mathbf{k}}}\mathbf{2}(kh) - \dot{\mathbf{B}}(kh)\bar{\mathbf{k}}\mathbf{2}(kh)\right]\Bigg)\bar{\mathbf{x}}2(kh) \tag{41}$$

So, (40) and (41) are the state determined for optimal input via first and second order Taylor series respectively. Further, the optimal input **u∗(*t*)** is calculated modifying (23) as,

$$\bar{\mathbf{u}}\mathbf{1}^*(t) = -\bar{\mathbf{k}}\mathbf{1}(t)\bar{\mathbf{x}}\mathbf{1}(t) \quad \text{and} \quad \bar{\mathbf{u}}\mathbf{2}^*(t) = -\bar{\mathbf{k}}\mathbf{2}(t)\bar{\mathbf{x}}\mathbf{2}(t) \tag{42}$$

calling the approximation of $\mathbf{u}^*(t)$ in first and second order Taylor domain be $\bar{\mathbf{u}}\mathbf{1}^*(t)$ and $\bar{\mathbf{u}}\mathbf{2}^*(t)$ respectively i. e., $\mathbf{u}^*(t) \triangleq \bar{\mathbf{u}}\mathbf{1}^*(t)$ and $\mathbf{u}^*(t) \triangleq \bar{\mathbf{u}}\mathbf{2}^*(t)$.

4. Numerical Example

Example 1.

Consider the linear non-homogeneous time varying system [3]

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & t\,e^{-t} \\ 0 & 0 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\mathbf{u}(t)\ ;\ \mathbf{x(0)} = \mathbf{x_0} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ with step input} \tag{43}$$

The time varying system given in this example is solved by using the recursive equations (19) and (20) subsequently. The solutions are shown in **Table 1** along with the exact samples [8]. For computation, let, T = 1 s. and m = 10, so that h = T/m = 0.1 s. The comparison of the exact solution with approximations is shown in figure 1(a). Decreasing the step size h, obviously improves the accuracy of approximation with increased computational burden. The accuracy can be improvement for different values of m (m=10, m=20, m=40) in the solutions via first and second order Taylor series and hence percentage error for state x_1 is shown in figure 1(b). Further, MISE using (21) for different m (m=10, m=20, m=40), shows improvement in Table 2.

Example 2

Consider the the time varying system [2, 7], $\dot{x}(t) = t\,x(t) + u(t)$ (44)

And quadratic performance index, $J = \frac{1}{2}\int_0^1 \left[x^2(t)+u^2(t)\right]dt$ (45)

Let, m=10 within $t \in[0, 1]$ s., where, T=1 s. So, $h = T/m$ = 0.1 s. Now, the approximate solution of optimal gains $\bar{k}1$ and $\bar{k}2$ is solved using equations (31-32) and is provided in the Table 3 along with the exact optimal gain [7] and percentage error. Figure 2 shows the solution of $k(t)$ via Taylor series and compares with the exact gain. In Table 4, the approximation of x and u calculated from equations (40), (42) and (41), (42) is shown using first and second order Taylor series respectively.

Table 1: Recursive solution of the state $x1$ and state $x2$ obtained via first order and second order Taylor approximation [7] compared with the exact solution (for T=1 s, m=10 and h=0.1 s).

Time (s.)	Pointwise solution of state $x1(t)$			Pointwise solution of state $x2(t)$		
	Exact data	Approximated data using		Exact data	Approximated data using	
		first order Taylor series	second order Taylor series		first order Taylor series	second order Taylor series
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1/10	1.0050	1.0000	1.0050	1.1000	1.1000	1.1000
2/10	1.0198	1.0100	1.0199	1.2000	1.2000	1.2000
3/10	1.0441	1.0296	1.0443	1.3000	1.3000	1.3000
4/10	1.0774	1.0585	1.0777	1.4000	1.4000	1.4000
5/10	1.1190	1.0960	1.1194	1.5000	1.5000	1.5000
6/10	1.1681	1.1415	1.1686	1.6000	1.6000	1.6000
7/10	1.2241	1.1942	1.2247	1.7000	1.7000	1.7000
8/10	1.2861	1.2533	1.2868	1.8000	1.8000	1.8000
9/10	1.3532	1.3180	1.3541	1.9000	1.9000	1.9000
10/10	1.4248	1.3875	1.4259	2.0000	2.0000	2.0000

Table 2: MISE for the approximation of state x_1 and state x_2 obtained via first order and second order Taylor series for different m and T=1 s.

number of recursion points (m) and length of intervals (h)	For states $\mathbf{x}(t)$	Approximation Error (MISE) using Taylor series of	
		First order	Second order
m=10; h=0.1 s.	State $x_1(t)$	10^{-3}x0.5581	10^{-5} x0.1163
	State $x_2(t)$	0	0
m=20; h=0.05s.	State $x_1(t)$	10^{-4}x0.2225	10^{-7}x0.2898
	State $x_2(t)$	0	0
m=40; h=0.025 s.	State $x_1(t)$	10^{-6}x0.7510	10^{-9}x0.8386
	State $x_2(t)$	0	0

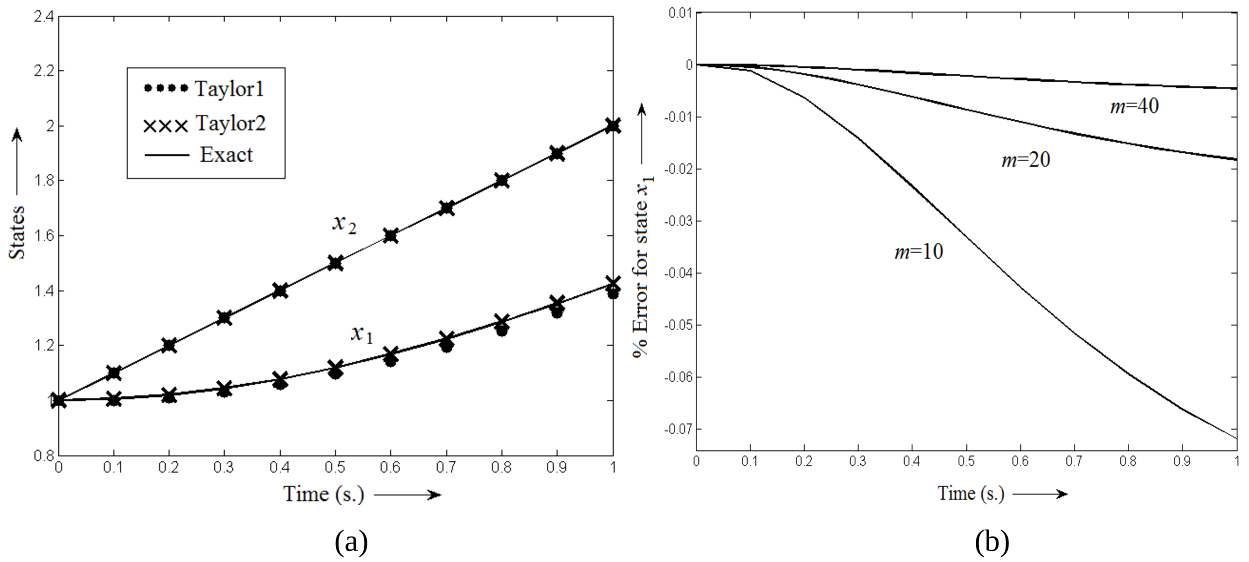

Fig. 1: (a)The exact solution along with first and second order approximation (Taylor1 and Taylor2) for T=1 s and m=10 and (b) The % error for state x_1 considering m=10, m=20, m=40.

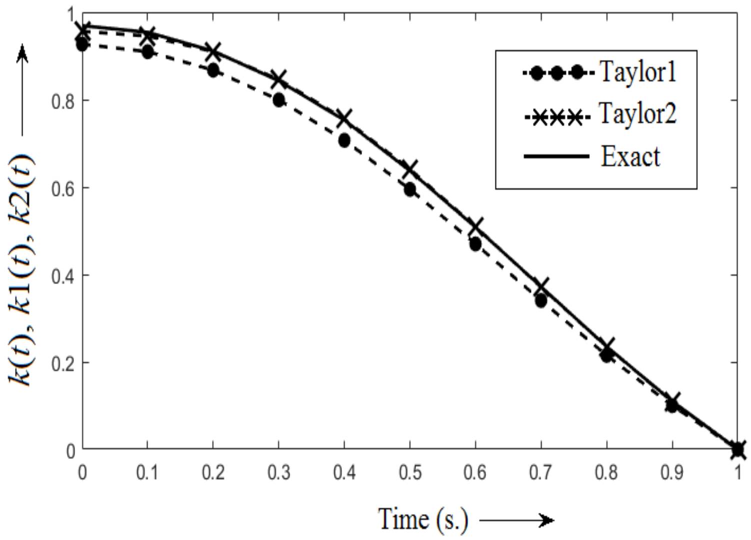

Fig. 2: The exact optimal gain k along with its approximation via (a) first order Taylor series (Taylor1) and second order Taylor series (Taylor2) for T=1 s and m=10

4. Conclusions

The proposition establishes recursive method using first and second order Taylor series. Although, higher order Taylor expansion may be opted, but, for practical realization, terms upto second derivative is good enough to choose. Conventionally, the state analysis is done via Taylor series [2-4] using operational matrix of integration of fixed dimension depending on h or m. In this respect, as this method does not involve operational matrices, it is not necessary to fix the interval length and hence one can easily change h or m during iteration performing a dynamic computation. This flexibility makes the algorithm more reasonable to choose.

Table 3: Recursive solution of the optimal gain $k(t)$ obtained via first order and second order Taylor approximation compared with the exact solution (for T=1 s, m=10 and h=0.1 s).

Time (s.)	Pointwise solution of optimal gain $k(t)$			Percentage error in	
	Exact data	Approximated data using		first order Taylor domain	second order Taylor domain
		first order Taylor series	second order Taylor series		
0	0.969	0.927	0.956	4.359	1.334
1/10	0.954	0.909	0.945	4.664	0.895
2/10	0.911	0.867	0.910	4.787	0.146
3/10	0.843	0.799	0.846	5.183	-0.377
4/10	0.753	0.707	0.755	6.127	-0.210
5/10	0.637	0.595	0.641	6.648	-0.570
6/10	0.508	0.470	0.509	7.531	-0.274
7/10	0.372	0.340	0.371	8.474	0.322
8/10	0.235	0.215	0.235	8.511	0.191
9/10	0.110	0.100	0.109	9.091	0.909
10/10	0.000	0.000	0.000	-	-

Table 4: Recursive solution of $x(t)$ and $u(t)$ with the exact solution (for T=1 s, m=10 and h=0.1 s).

Time (s.)	Solution of state $x(t)$ using		Solution of input $u(t)$ using	
	first order Taylor series	second order Taylor series	first order Taylor series	second order Taylor series
0	1.000	1.000	-0.927	-0.956
1/10	0.907	0.909	-0.825	-0.860
2/10	0.834	0.837	-0.723	-0.761
3/10	0.778	0.781	-0.622	-0.661
4/10	0.739	0.741	-0.523	-0.560
5/10	0.717	0.718	-0.426	-0.460
6/10	0.710	0.712	-0.333	-0.362
7/10	0.719	0.722	-0.245	-0.268
8/10	0.745	0.750	-0.160	-0.176
9/10	0.789	0.798	-0.079	-0.087
10/10	0.852	0.868	0.000	0.000

Moreover, this procedure being recursive, avoids the complexity of handling large matrices and its inversion too unlike Kronecker product approach. The memory requirement and time consumption during computation is reduced considerably in this context. In optimal control problem for the method proposed by Radhoush et al [7] the approximated value of k is nearly same with the exact data using Chebyshev wavelet, but it approaches through Kronecker product and has complexity in computation, whereas this method results the samples of optimal gain with a little bit of error more but still proves to be stronger because of its simplicity with less computation time. Even, the degree of Chebyshev polynomial [7] had been chosen as 4. Further compared to another method [2] where Taylor series has been employed with higher order and produces still more erroneous results, this algorithm only upto second order derivative term is way better in this sense. So, the recursive approach with only first and second order Taylor series is more attractive because of its straightforward nature and reliability as well.

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