Adding up (2.10) and (2.11), we have

$q(gx_n, gx_{n+1}) + q(gx_{n+1}, gx_n)(k(gx_0) + r(gx_0) + t(gx_0))[q(gx_{n-1}, gx_n) + q(gx_n, gx_{n-1})] + (l(gx_0) + t(gx_0))[(q(gx_{n+1}, gx_n) + q(gx_n, gx_{n+1})]$. (2.12)

Now, set $v_n = q(gx_n, gx_{n+1}) + q(gx_{n+1}, gx_n)$ in (2.12), we have

$$v_n \preccurlyeq \left(k(gx_0) + r(gx_0) + t(gx_0)\right)v_{n-1} + (l(gx_0) + t(gx_0))v_n.$$

So $v_n \preccurlyeq \mu v_{n-1}$ for all $n \geq 1$ with $\mu = \frac{k(gx_0)+r(gx_0)+t(gx_0)}{1-l(gx_0)-t(gx_0)} < 1$.

Since $(k + l + r + 2t)(x) < 1$ for all $x \in X$.

Continuing this process, we get $v_n \preccurlyeq \mu^n v_0$ for $n = 0, 1, 2, \ldots$...

Rest of the proof of this theorem is similar as the Theorem 2.1.

Example 2.6. Let $E = \mathbb{R}$ and $P = \{x \in E: x \geq 0\}$. Let $X = [0,1]$ and define a mapping $d: X \text{ x } X \to E$ by $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a cone metric space. Define a mapping $q: X \text{ x } X \to E$ by $q(x, y) = 2d(x, y)$ for all $x, y \in X$. Then q is a c-Distance. In fact, $(q_1) - (q_3)$ are immediate.

Let $c \in E$ with $0 \ll c$ put $e = \frac{c}{2}$. If $q(z, x) \ll e$ and $q(z, y) \ll e$, then we have $d(x, y) \leq 2d(x, y) = 2|x - y| \leq 2|x - z| + 2|z - y| = q(z, x) + q(z, y) \ll e + e = c$.

This shows that (q_4) holds. Therefore q is a c-Distance.

Let $f, g: X \to X$ defined by $g(x) = x$ and $f(x) = \frac{x^2}{16}$ for all $x \in X$.

Take mappings $k, l, r, t: X \to [0,1)$ by $k(x) = \frac{x+1}{16}$, $r(x) = \frac{2x+3}{16}$, $l(x) = \frac{3x+2}{16}$, $t(x) = \frac{x}{16}$ for all $x \in X$. Observe that

(i) $k(fx) = (\frac{x^2}{16} + 1)/16 = \frac{1}{16}\left(\frac{x^2}{16} + 1\right) \leq \frac{1}{16}(x + 1) = k(x) = k(gx)$.

$(ii) r(fx) = (2(\frac{x^2}{16}) + 3)/16 = \frac{1}{16}\left(\frac{2x^2}{16} + 3\right) \leq \frac{1}{16}(2x + 3) = r(x) = r(gx)$.

$(iii) l(fx) = (3(\frac{x^2}{16}) + 2)/16 = \frac{1}{16}\left(\frac{3x^2}{16} + 2\right) \leq \frac{1}{16}(3x + 2) = l(x) = l(gx)$.

(iv) $t(fx) = (\frac{x^2}{16})/16 = \frac{1}{16}\left(\frac{x^2}{16}\right) \leq \frac{1}{16}(x) = t(x) = t(gx)$.

(v) $(k + l + r + 2t)(x) = \left(\frac{x+1}{16}\right) + \left(\frac{3x+2}{16}\right) + \left(\frac{2x+3}{16}\right) + 2(\frac{x}{16}) = \left(\frac{8x+6}{16}\right) < 1$ for all $x \in X$.

(vi) for all $x, y \in X$, we have

$$q(fx, fy) = 2\left|\frac{x^2}{16} - \frac{y^2}{16}\right| \preccurlyeq \frac{2|x+y||x-y|}{16} = \left(\frac{x+y}{16}\right)2|x - y|$$
$$\preccurlyeq k(x)q(x, y) = k(gx)q(gx, gy)$$
$$\preccurlyeq k(gx)q(gx, gy) + l(gx)q(fy, gy) + r(gx)q(fx, gx) + t(gx)[q(fx, gy) + q(fy, gx)].$$

Therefore, all the conditions of Theorem 2.5 are satisfied. Hence f and g have a common fixed point in X. This common fixed point is $x = 0$.

3. CONCLUSION

In this paper we develop and generalize the common fixed point theorems on c-Distance of Kaewkhao et al. [5], Rahimi et al. [7] and Young et al. [17]. One illustrative example is also furnished to highlight the realized improvements.

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