AMSE JOURNALS-2016-Series: Advances A; Vol. 53; N°2; pp 48-59

Submitted July 2016; Revised Oct. 15, 2016, Accepted Dec. 10, 2016

On the Cauchy Problem for Three-Dimensional Generalized Zakharov Equations

Shujun You, Xiaoqi Ning*

Department of Mathematics, Huaihua University, Huaihua 418008, China,
*Corresponding author Email: nxq035@163.com

Abstract

This paper considers the existence of the generalized solution to the Cauchy problem for a class of generalized Zakharov equation in three dimensions. By a priori integral estimates and Galerkin method, one has the existence of the global generalized solution to the problem.

Keywords: Generalized Zakharov equations, Cauchy problem, generalized solution

1. Introduction

In the past decade, the Zakharov system was studied by many authors [3-7,10-12]. Morris, Kara and Biswas study the Zakharov equation with power law nonlinearity. The traveling wave hypothesis is applied to obtain the 1-soliton solution of this equation. The multiplier method from Lie symmetries is subsequently utilized to obtain the conservation laws of the equations [10]. Bhrawy, Abdelkawy and Biswas study the Zakharov equation by the aid of Jacobi's elliptic function expansion method and exact periodic solutions are extracted [11]. Agafontsev and Zakharov study numerically the statistics of waves for generalized one-dimensional Nonlinear Schrödinger equation that takes into account focusing six-wave interactions, dumping and pumping terms [12].

Special interest was recently devoted to quantum corrections to the Zakharov equations for Langmuir waves in a plasma. First considered in one space dimension [1]. The model was then extended to two and three dimensions [2].

$$iE_t - \alpha\nabla\times(\nabla\times E) + \nabla(\nabla\cdot E) = nE + \Gamma\nabla\Delta(\nabla\cdot E),$$
$$n_{tt} - \Delta n = \Delta|E|^2 - \Gamma\Delta^2 n.$$

where $E:\mathbb{R}^{d+1}\to\mathbb{R}^d$ is the slowly varying amplitude of the high-frequency electric field, and $n:\mathbb{R}^{d+1}\to\mathbb{R}$ denotes the fluctuation of the ion-density from its equilibrium. the parameter α defined as the square ratio of the light speed and the electron Fermi velocity is usually large. In contrast, the coefficient Γ that measures the influence of quantum effects is usually very small [9].

the quantum Zakharov system was studied by some authors [3-7]. In this paper, we are interested in studying the following generalized modified Zakharov system in dimension three.

$$iE_t-\alpha\nabla\times(\nabla\times E)+\nabla(\nabla\cdot E)=nE+\Gamma\nabla\Delta(\nabla\cdot E)+f(|E|^2)E, \tag{1}$$

$$n_{tt}-\Delta n=\Delta|E|^2-\Gamma\Delta^2 n. \tag{2}$$

with initial data

$$E(x,0)=E_0(x),\quad n(x,0)=n_0(x),\quad n_t(x,0)=n_1(x). \tag{3}$$

where $E=(E_1,E_2,E_3):\mathbb{R}^{3+1}\to\mathbb{R}^3$, $n:\mathbb{R}^{3+1}\to\mathbb{R}$, $x=(x_1,x_2,x_3)\in\mathbb{R}^3$.

Now we state the main results of the paper.

Theorem 1. Suppose that

(i) $E_0(x)\in H^2(\mathbb{R}^3),\ n_0(x)\in H^1(\mathbb{R}^3),\ n_1(x)\in H^{-1}(\mathbb{R}^3).$

(ii) $f(\xi)\in C(\mathbb{R}),\ |f(\xi)|\le M\xi^\gamma,\ |f(\xi)-f(\zeta)|\le L|\xi-\zeta|$. Where $M>0$, $0\le\gamma<\frac{2}{3}$, $L>0$.

Then there exists global generalized solution of the initial problem (1)-(3).

$$\begin{aligned}&E(x,t)\in L^\infty(\mathbb{R}^+;H^1)\cap W^{1,\infty}(\mathbb{R}^+;H^{-2}),\\&n(x,t)\in L^\infty(\mathbb{R}^+;H^1)\cap W^{1,\infty}(\mathbb{R}^+;H^{-1}),\\&n_t(x,t)\in L^\infty(\mathbb{R}^+;H^{-1})\cap W^{1,\infty}(\mathbb{R}^+;H^{-3}).\end{aligned}$$

To study generalized solution of the system (1)-(3), we transform it into the following form (notice that $\nabla(\nabla\cdot E)=\Delta E+\nabla\times(\nabla\times E)$)

$$iE_t-(\alpha-1)\nabla\times(\nabla\times E)+\Delta E=nE+\Gamma\nabla\Delta(\nabla\cdot E)+f(|E|^2)E, \tag{4}$$

$$n_t+\nabla\cdot\varphi=0, \tag{5}$$

$$\varphi_t=-\nabla(n+|E|^2)+\Gamma\nabla\Delta n. \tag{6}$$

with initial data

$$E(x,0)=E_0(x),\ n(x,0)=n_0(x),\ \varphi(x,0)=\varphi_0(x). \tag{7}$$

where φ_0 satisfying $-\nabla\cdot\varphi_0=n_1$.

For the sake of convenience of the following contexts, we set some notations. For $1 \le q \le \infty$, we denote $L^q(\mathbb{R}^d)$ the space of all q th power integrable functions in $\mathbb{R}^d$ equipped with norm $\|\cdot\|_{L^q(\mathbb{R}^d)}$. We write $H^s(\mathbb{R}^d)$ instead of the Sobolev space $H^{s,2}(\mathbb{R}^d)$. Let $(f,g) = \int_{\mathbb{R}^n} f(x)\cdot\overline{g(x)}dx$, where $\overline{g(x)}$ denotes the complex conjugate function of $g(x)$. And we use C to represent various constants that can depend on initial data. In Section 2, we establish a priori estimations. In Section 3, we state the existence of global generalized solution.

2. *A priori* estimations

Taking the inner product of (4) and E, and then taking the imaginary part, we get

$$\|E(x,t)\|^2_{L^2(\mathbb{R}^3)} = \|E_0\|^2_{L^2(\mathbb{R}^3)}$$

Lemma 1. Suppose that $E_0(x) \in H^2(\mathbb{R}^3),\ n_0(x) \in H^1(\mathbb{R}^3),\ \varphi_0(x) \in L^2(\mathbb{R}^3)$ and $f(\xi) \in C(\mathbb{R})$. Then for the solution of problem (4)-(7) we have $\mathrm{A}(t) = \mathrm{A}(0),$ where

$$\begin{aligned}\mathrm{A}(t) = \|\nabla E\|^2_{L^2} + (\alpha-1)\|\nabla\times E\|^2_{L^2} + \int n\,|E|^2\,dx + \Gamma\|\nabla(\nabla\cdot E)\|^2_{L^2} \\ + \int\int_0^{|E|^2} f(\xi)d\xi dx + \frac{\Gamma}{2}\|\nabla n\|^2_{L^2} + \frac{1}{2}\|n\|^2_{L^2} + \frac{1}{2}\|\varphi\|^2_{L^2}.\end{aligned}$$

Proof. Taking the inner product of (4) and E_t. Since

$$\mathrm{Re}\big(-(\alpha-1)\nabla\times(\nabla\times E),\, E_t\big) = -\frac{\alpha-1}{2}\frac{d}{dt}\|\nabla\times E\|^2_{L^2},$$

$$\mathrm{Re}\big(\Delta E,\, E_t\big) = -\frac{1}{2}\frac{d}{dt}\|\nabla E\|^2_{L^2},$$

$$\mathrm{Re}\big(nE,\, E_t\big) = \frac{1}{2}\int n\Big(|E|^2\Big)_t dx = \frac{1}{2}\frac{d}{dt}\int n|E|^2 dx - \frac{1}{2}\int n_t\,|E|^2 dx,$$

$$\mathrm{Re}\big(\Gamma\nabla\Delta(\nabla\cdot E),\, E_t\big) = \Gamma\,\mathrm{Re}\big(\nabla(\nabla\cdot E), \nabla(\nabla\cdot E_t)\big) = \frac{\Gamma}{2}\frac{d}{dt}\|\nabla(\nabla\cdot E)\|^2_{L^2},$$

$$\mathrm{Re}\Big(f(|E|^2)E,\, E_t\Big) = \frac{1}{2}\int f(|E|^2)(|E|^2)_t\,dx = \frac{1}{2}\frac{d}{dt}\int\int_0^{|E|^2} f(\xi)d\xi dx.$$

Thus we get

$$\begin{aligned}\frac{d}{dt}\Big[\|\nabla E\|^2_{L^2} + (\alpha-1)\|\nabla\times E\|^2_{L^2} + \int n|E|^2 dx\Big] \\ + \frac{d}{dt}\Big[\Gamma\|\nabla(\nabla\cdot E)\|^2_{L^2} + \int\int_0^{|E|^2} f(\xi)d\xi dx\Big] = \int n_t\,|E|^2 dx.\end{aligned} \tag{8}$$

From (5) and (6), we obtain

$$\int n_t |E|^2\, dx = -\int \nabla\cdot\varphi |E|^2\, dx = \int \varphi\cdot\nabla |E|^2\, dx$$

$$= \int \varphi\cdot(\Gamma\nabla\Delta n - \nabla n - \varphi_t)dx = \int \nabla\cdot\varphi(-\Gamma\Delta n + n)dx - \frac{1}{2}\frac{d}{dt}\|\varphi\|_{L^2}^2 \tag{9}$$

$$= \int n_t(\Gamma\Delta n - n)dx - \frac{1}{2}\frac{d}{dt}\|\varphi\|_{L^2}^2 = -\frac{1}{2}\frac{d}{dt}\left[\Gamma\|\nabla n\|_{L^2}^2 + \|n\|_{L^2}^2 + \|\varphi\|_{L^2}^2\right].$$

Combining inequality (8) with (9) we obtain

$$\frac{d}{dt}\left[\|\nabla E\|_{L^2}^2 + (\alpha-1)\|\nabla\times E\|_{L^2}^2 + \int n|E|^2\, dx + \Gamma\|\nabla(\nabla\cdot E)\|_{L^2}^2\right]$$

$$+\frac{d}{dt}\left[\int\int_0^{|E|^2} f(\xi)d\xi dx + \frac{\Gamma}{2}\|\nabla n\|_{L^2}^2 + \frac{1}{2}\|n\|_{L^2}^2 + \frac{1}{2}\|\varphi\|_{L^2}^2\right] = 0.$$

Lemma 1 is proved.

Lemma 2. Suppose that

(i) $E_0(x)\in H^2(\mathbb{R}^3),\ n_0(x)\in H^1(\mathbb{R}^3),\ \varphi_0(x)\in L^2(\mathbb{R}^3)$.

(ii) $f(\xi)\in C(\mathbb{R}),\ |f(\xi)|\le M\xi^\gamma$. Where $M>0,\ 0\le\gamma<\frac{2}{3}$.

Then we have

$$\|\nabla E\|_{L^2}^2 + \|\nabla\times E\|_{L^2}^2 + \|\nabla(\nabla\cdot E)\|_{L^2}^2 + \|\nabla n\|_{L^2}^2 + \|n\|_{L^2}^2 + \|\varphi\|_{L^2}^2 \le C.$$

Proof. By Hölder inequality, Young inequality and Lemma 1 we have

$$\int n|E|^2\, dx \le \|n\|_{L^4}\|E\|_{L^{\frac{8}{3}}}^2 \le C\|\nabla n\|_{L^2}^{\frac{3}{4}}\|n\|_{L^2}^{\frac{1}{4}}\|\nabla E\|_{L^2}^{\frac{3}{4}}\|E\|_{L^2}^{\frac{5}{4}}$$

$$\le \frac{\Gamma}{4}\|\nabla n\|_{L^2}^2 + C\|n\|_{L^2}^{\frac{2}{5}}\|\nabla E\|_{L^2}^{\frac{6}{5}} \le \frac{\Gamma}{4}\|\nabla n\|_{L^2}^2 + \frac{1}{4}\|n\|_{L^2}^2 + C\|\nabla E\|_{L^2}^{\frac{3}{2}} \tag{10}$$

$$\le \frac{\Gamma}{4}\|\nabla n\|_{L^2}^2 + \frac{1}{4}\|n\|_{L^2}^2 + \frac{1}{4}\|\nabla E\|_{L^2}^2 + C.$$

and noticing that $f(\xi)\in C(\mathbb{R}),\ |f(\xi)|\le M\xi^\gamma$, we get

$$\int\int_0^{|E|^2} f(\xi)d\xi dx \le \int\int_0^{|E|^2} M\xi^\gamma d\xi dx = \frac{M}{\gamma+1}\int |E|^{2(\gamma+1)}\, dx \tag{11}$$

Using Gagliardo-Nirenberg inequality and noticing that $0\le\gamma<\frac{2}{3}$, we write

$$\frac{M}{\gamma+1}\int |E|^{2(\gamma+1)}\, dx \le C\|\nabla E\|_{L^2}^{3\gamma}\|E\|_{L^2}^{2-\gamma} \le \frac{1}{4}\|\nabla E\|_{L^2}^2 + C. \tag{12}$$

Note that from Lemma 1 and eq. (10)-(12), one has

$$\frac{1}{2}\|\nabla E\|_{L^2}^2+(\alpha-1)\|\nabla\times E\|_{L^2}^2+\Gamma\|\nabla(\nabla\cdot E)\|_{L^2}^2$$
$$+\frac{\Gamma}{4}\|\nabla n\|_{L^2}^2+\frac{1}{4}\|n\|_{L^2}^2+\frac{1}{2}\|\varphi\|_{L^2}^2\le|\mathrm{A}(0)|+C.$$

Since α is larger than 1, we thus get Lemma 2.

Lemma 3. Suppose that

(i) $E_0(x)\in H^2(\mathbb{R}^3),\ n_0(x)\in H^1(\mathbb{R}^3),\ \varphi_0(x)\in L^2(\mathbb{R}^3).$

(ii) $f(\xi)\in C(\mathbb{R}),\ |f(\xi)|\le M\xi^\gamma$. Where $M>0,\ 0\le\gamma<\frac{2}{3}.$

Then we have

$$\|E_t\|_{H^{-2}}+\|n_t\|_{H^{-1}}+\|\varphi_t\|_{H^{-2}}\le C.$$

Proof. Taking the inner product of eq. (4) and V, (5) and v, (6) and Φ, it follows that

$$\left(iE_t-(\alpha-1)\nabla\times(\nabla\times E)+\Delta E,V\right)=\left(nE+\Gamma\nabla\Delta(\nabla\cdot E)+f(|E|^2)E,V\right). \tag{13}$$

$$\left(n_t+\nabla\cdot\varphi,v\right)=0, \tag{14}$$

$$\left(\varphi_t,\Phi\right)=\left(-\nabla(n+|E|^2)+\Gamma\nabla\Delta n,\Phi\right). \tag{15}$$

where $\forall\ v,v_i\in H_0^2\ \ (i=1,2,3)$, $V=(v_1,v_2,v_3)$.

By Hölder inequality, it follows from eq. (13) that

$$\begin{aligned}\left|\left(E_t,V\right)\right|&\le\left|\left((\alpha-1)\nabla\times\left(\nabla\times E\right),V\right)\right|+\left|\left(\Delta E,V\right)\right|+\left|\left(nE,V\right)\right|\\&\quad+\left|\left(\Gamma\nabla\left[\Delta\left(\nabla\cdot E\right)\right],V\right)\right|+\left|\left(f(|E|^2)E,V\right)\right|\\&=(\alpha-1)\left|\left(\nabla\times E,\nabla\times V\right)\right|+\left|\left(\nabla E,\nabla V\right)\right|+\left|\left(nE,V\right)\right|\\&\quad+\Gamma\left|\left(\nabla\left(\nabla\cdot E\right),\nabla\left(\nabla\cdot V\right)\right)\right|+\left|\left(f(|E|^2)E,V\right)\right|\\&\le(\alpha-1)\|\nabla\times E\|_{L^2}\|\nabla\times V\|_{L^2}+\|\nabla E\|_{L^2}\|\nabla V\|_{L^2}+\|n\|_{L^4}\|E\|_{L^4}\|V\|_{L^2}+\\&\quad+\Gamma\left\|\nabla\left(\nabla\cdot E\right)\right\|_{L^2}\left\|\nabla\left(\nabla\cdot V\right)\right\|_{L^2}+M\|E\|_{L^{2(2\gamma+1)}}^{2\gamma+1}\|V\|_{L^2}.\end{aligned} \tag{16}$$

By Gagliardo-Nirenberg inequality, we know that

$$\|E\|_{L^4}\le C\|\nabla E\|_{L^2}^{\frac{3}{4}}\|E\|_{L^2}^{\frac{1}{4}}\le C,$$

$$\|n\|_{L^4}\le C\|\nabla n\|_{L^2}^{\frac{3}{4}}\|n\|_{L^2}^{\frac{1}{4}}\le C,$$

$$\|E\|_{L^{2(2\gamma+1)}}^{2\gamma+1}\le C\|\nabla E\|_{L^2}^{3\gamma}\|E\|_{L^2}^{1-\gamma}\le C,$$

Hence from (16) we get

$$\left|\left(E_t,V\right)\right|\le C\|V\|_{H_0^2}. \tag{17}$$

Using Hölder inequality, from eq. (14), there is

$$\left|(n_t, v)\right| = \left|(\nabla\cdot\varphi, v)\right| = \left|(\varphi, \nabla v)\right| \le \|\varphi\|_{L^2}\|\nabla v\|_{L^2} \le C\|v\|_{H_0^1}, \tag{18}$$

From eq. (15) and Hölder inequality, we have

$$\begin{aligned}\left|(\varphi_t, V)\right| &= \left|(\nabla n, V)\right| + \left|(\nabla |E|^2, V)\right| + \left|(\Gamma\nabla\Delta n, V)\right| \\ &\le \|\nabla n\|_{L^2}\|V\|_{L^2} + \left|(|E|^2, \nabla\cdot V)\right| + \Gamma\left|(\nabla n, \Delta V)\right| \\ &\le \|\nabla n\|_{L^2}\|V\|_{L^2} + \|E\|_{L^4}^2\|\nabla\cdot V\|_{L^2} + \Gamma\|\nabla n\|_{L^2}\|\Delta V\|_{L^2} \\ &\le C\|V\|_{H_0^2}.\end{aligned} \tag{19}$$

Hence from (17)-(19), one obtain Lemma 3.

3. The existence of generalized solution

In this section, we formulate the proof of Theorem 1. First we give the definition of generalized solution for problem (4)-(7).

Definition 1. The functions $E(x,t) \in L^\infty(\mathbb{R}^+; H^1) \cap W^{1,\infty}(\mathbb{R}^+; H^{-2})$, $n(x,t) \in L^\infty(\mathbb{R}^+; H^1) \cap W^{1,\infty}(\mathbb{R}^+; H^{-1})$, $\varphi(x,t) \in L^\infty(\mathbb{R}^+; L^2) \cap W^{1,\infty}(\mathbb{R}^+; H^{-2})$, are called generalized solution of problem (4)-(7), if they satisfy the following integral equality

$$\begin{aligned}&(iE_{mt}, v) + (\alpha-1)\sum_{\substack{\tau\ne m\\ \tau\in\{1,2,3\}}}\left(\frac{\partial E_\tau}{\partial x_m}, \frac{\partial v}{\partial x_\tau}\right) - (\alpha-1)\sum_{\substack{\tau\ne m\\ \tau\in\{1,2,3\}}}\left(\frac{\partial E_m}{\partial x_\tau}, \frac{\partial v}{\partial x_\tau}\right) - (\nabla E_m, \nabla v) \\ &\quad = (nE_m, v) + \Gamma\left(\nabla(\nabla\cdot E), \nabla\left(\frac{\partial v}{\partial x_m}\right)\right) + \left(f(|E|^2)E_m, v\right), \quad m = 1,2,3, \\ &(n_t + \nabla\cdot\varphi, v) = 0, \\ &(\varphi_{\lambda t}, v) = \left(-\frac{\partial(n+|E|^2)}{\partial x_\lambda} + \Gamma\frac{\partial(\Delta n)}{\partial x_\lambda}, v\right), \quad \lambda = 1,2,3.\end{aligned}$$

with initial data

$$E|_{t=0} = E_0(x), \ n|_{t=0} = n_0(x), \ \varphi|_{t=0} = \varphi_0(x),$$

Next, we give two lemmas recalled in [8].

Lemma 4. Let B_0, B, B_1 be three reflexive Banach spaces and assume that the embedding $B_0 \to B$ is compact. Let

$$W = \left\{V \in L^{p_0}((0,T); B_0), \frac{\partial V}{\partial t} \in L^{p_1}((0,T); B_1)\right\}, \quad T < \infty, 1 < p_0, p_1 < \infty.$$

W is a Banach space with norm

$$\|V\|_W = \|V\|_{L^{p_0}((0,T);B_0)} + \|V_t\|_{L^{p_1}((0,T);B_1)}.$$

Then the embedding $W \to L^{p_0}((0,T);B)$ is compact.

Lemma 5. Let Ω be an open set of $\mathbb{R}^n$ and let $g, g_\varepsilon \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, such that

$$g_\varepsilon \to g \quad \text{a.e. in } \Omega \qquad \text{and} \qquad \|g_\varepsilon\|_{L^p(\Omega)} \le C.$$

Then $g_\varepsilon \to g$ weakly in $L^p(\Omega)$.

Now, one can estimate the following theorem.

Theorem 2. Suppose that

(i) $E_0(x) \in H^2(\mathbb{R}^3)$, $n_0(x) \in H^1(\mathbb{R}^3)$, $\varphi_0(x) \in L^2(\mathbb{R}^3)$.

(ii) $f(\xi) \in C(\mathbb{R})$, $|f(\xi)| \le M\xi^\gamma$, $|f(\xi) - f(\zeta)| \le L|\xi - \zeta|$. Where $M > 0$, $0 \le \gamma < \frac{2}{3}$, $L > 0$.

Then there exists global generalized solution of the initial value problem (4)-(7).

$$\begin{aligned} &E(x,t) \in L^\infty(\mathbb{R}^+; H^1) \cap W^{1,\infty}(\mathbb{R}^+; H^{-2}), \\ &n(x,t) \in L^\infty(\mathbb{R}^+; H^1) \cap W^{1,\infty}(\mathbb{R}^+; H^{-1}), \\ &\varphi(x,t) \in L^\infty(\mathbb{R}^+; L^2) \cap W^{1,\infty}(\mathbb{R}^+; H^{-2}). \end{aligned}$$

Proof. By using Galerkin method, choose the basic periodic functions $\{\omega_j(x)\}$ as follows:

$$-\Delta\omega_j(x) = \lambda_j\omega_j(x), \ \omega_j(x) \in H_0^2(\Omega), \ j = 1,2,\cdots,l.$$

The approximate solution of problem (4)-(7) can be written as

$$E^l(x,t) = \sum_{j=1}^{l} \alpha_j^l(t)\omega_j(x), \quad \varphi^l(x,t) = \sum_{j=1}^{l} \beta_j^l(t)\omega_j(x), \quad n^l(x,t) = \sum_{j=1}^{l} \gamma_j^l(t)\omega_j(x),$$

where

$$E^l = \left(E_1^l, E_2^l, E_3^l\right), \quad \alpha_j^l(t) = (\alpha_{j1}^l(t), \alpha_{j2}^l(t), \alpha_{j3}^l(t)).$$

$$\varphi^l = \left(\varphi_1^l, \varphi_2^l, \varphi_3^l\right), \quad \beta_j(t) = (\beta_{j1}(t), \beta_{j2}(t), \beta_{j3}(t)).$$

and Ω is a 3-dimensional cube with $2D$ in each direction, that is, $\overline{\Omega} = \{x = (x_1, x_2, x_3) \,\|\, x_i | \le 2D, i = 1,2,3\}$. According to Galerkin's method, these undetermined coefficients $\alpha_j^l(t)$, $\beta_j^l(t)$ and $\gamma_j^l(t)$ need to satisfy the following initial value problem of the system of ordinary differential equations

$$\left(iE_{mt}^{l},\omega_{\kappa}\right)+(\alpha-1)\sum_{\substack{\nu\neq m\\ \nu\in\{1,2,3\}}}\left(\frac{\partial E_{\nu}^{l}}{\partial x_{m}},\frac{\partial\omega_{\kappa}}{\partial x_{\nu}}\right)-(\alpha-1)\sum_{\substack{\nu\neq m\\ \nu\in\{1,2,3\}}}\left(\frac{\partial E_{m}^{l}}{\partial x_{\nu}},\frac{\partial\omega_{\kappa}}{\partial x_{\nu}}\right)-\left(\nabla E_{m}^{l},\nabla\omega_{\kappa}\right)$$
$$=\left(n^{l}E_{m}^{l},\omega_{\kappa}\right)+\Gamma\left(\nabla\left(\nabla\cdot E^{l}\right),\nabla\left(\frac{\partial\omega_{\kappa}}{\partial x_{m}}\right)\right)+\left(f(|E^{l}|^{2})E_{m}^{l},\omega_{\kappa}\right),\quad m=1,2,3, \tag{20}$$

$$\left(n_{t}^{l}+\nabla\cdot\varphi^{l},\omega_{\kappa}\right)=0,\quad \kappa=1,2,\cdots,l, \tag{21}$$

$$\left(\varphi_{\lambda t}^{l},\omega_{\kappa}\right)=\left(-\frac{\partial(n^{l}+|E^{l}|^{2})}{\partial x_{\lambda}}+\Gamma\frac{\partial\left(\Delta n^{l}\right)}{\partial x_{\lambda}},\omega_{\kappa}\right),\quad \lambda=1,2,3, \tag{22}$$

with initial data

$$E^{l}|_{t=0}=E_{0}^{l}(x),\quad n^{l}|_{t=0}=n_{0}^{l}(x),\quad \varphi^{l}|_{t=0}=\varphi_{0}^{l}(x), \tag{23}$$

Suppose

$$E_{0}^{l}(x)\xrightarrow{H^{2}}E_{0}(x),\quad n_{0}^{l}(x)\xrightarrow{H^{1}}n_{0}(x),\quad \varphi_{0}^{l}(x)\xrightarrow{L^{2}}\varphi_{0}(x),\quad l\to\infty.$$

Similarly to the proof of Lemma 1-3, for the solution $E^{l}(x,t)$, $n^{l}(x,t)$, $\varphi^{l}(x,t)$ of problem (20)-(23), we can establish the following estimations

$$\left\|\nabla\times E^{l}\right\|_{L^{2}}^{2}+\left\|E^{l}\right\|_{H^{1}}^{2}+\left\|\nabla\left(\nabla\cdot E^{l}\right)\right\|_{L^{2}}^{2}+\left\|\varphi^{l}\right\|_{L^{2}}^{2}+\left\|n^{l}\right\|_{H^{1}}^{2}\leq C, \tag{24}$$

$$\left\|E_{t}^{l}\right\|_{H^{-2}}+\left\|\varphi_{t}^{l}\right\|_{H^{-2}}+\left\|n_{t}^{l}\right\|_{H^{-1}}\leq C. \tag{25}$$

where the constant C is independent of l and D. By compact argument, some subsequence of $\left(E^{l},n^{l},\varphi^{l}\right)$, also labeled by l, has a weak limit $\left(E,n,\varphi\right)$. More precisely

$$E^{l}\to E\quad\text{in}\quad L^{\infty}(\mathbb{R}^{+};H^{1})\quad\text{weakly star}, \tag{26}$$

$$n^{l}\to n\quad\text{in}\quad L^{\infty}(\mathbb{R}^{+};H^{1})\quad\text{weakly star}, \tag{27}$$

$$\varphi^{l}\to\varphi\quad\text{in}\quad L^{\infty}(\mathbb{R}^{+};L^{2})\quad\text{weakly star}.$$

Eq. (25) imply that

$$E_{t}^{l}\to E_{t}\quad\text{in}\quad L^{\infty}(\mathbb{R}^{+},H^{-2})\quad\text{weakly star}, \tag{28}$$

$$n_{t}^{l}\to n_{t}\quad\text{in}\quad L^{\infty}(\mathbb{R}^{+},H^{-1})\quad\text{weakly star},$$

$$\varphi_{t}^{l}\to\varphi_{t}\quad\text{in}\quad L^{\infty}(\mathbb{R}^{+},H^{-2})\quad\text{weakly star}.$$

Moreover, let us note that the following maps are continuous.

$$H^{1}(\mathbb{R}^{3})\to L^{4}(\mathbb{R}^{3}),\quad u\mapsto u,$$

$$H^{1}(\mathbb{R}^{3})\times H^{1}(\mathbb{R}^{3})\to L^{2}(\mathbb{R}^{3}),\quad (u,v)\mapsto uv.$$

It then follows from eq. (26) and (27) that

$$\left|E^l\right|^2 \to w \quad \text{in} \quad L^\infty(\mathbb{R}^+, L^2) \quad \text{weakly star,} \tag{29}$$

$$n^l E^l \to z \quad \text{in} \quad L^\infty(\mathbb{R}^+, L^2) \quad \text{weakly star.} \tag{30}$$

First, we prove $w = |E|^2$. Let Ω be any bounded subdomain of $\mathbb{R}^3$. We notice that

the embedding $H^1(\Omega) \to L^4(\Omega)$ is compact,

and for any Banach space X,

the embedding $L^\infty(\mathbb{R}^+, X) \to L^2(0,T;X)$ is continuous.

Hence, according to eq. (26), (28) and Lemma 4, applied to $B_0 = H^1(\Omega),\ B = L^4(\Omega),\ B_1 = H^{-2}(\Omega)$, and says that some subsequence of $E^l|_\Omega$ (also labeled by l) converges strongly to $E|_\Omega$ in $L^2(0,T;L^4(\Omega))$. So we can assume that

$$E^l \to E \quad \text{strongly in} \quad L^2(0,T;L^4_{loc}(\Omega)), \tag{31}$$

and thus

$$E^l \to E \quad \text{a.e. in} \quad [0,T]\times\Omega.$$

Then, using Lemma 5 and eq. (29) imply that $w = |E|^2$.

Second, we prove $z = nE$. Let ψ be some test function in $L^2(0,T;H^1),\ \operatorname{supp}\psi \subset \Omega \subset \mathbb{R}^3$.

$$\int_0^T \int_{\mathbb{R}^3} (n^l E^l - nE)\psi\, dxdt = \int_0^T \int_\Omega n^l \left(E^l - E\right)\psi\, dxdt + \int_0^T \int_\Omega \left(n^l - n\right)E\psi\, dxdt.$$

On one hand

$$\left|\int_0^T \int_\Omega n^l \left(E^l - E\right)\psi\, dxdt\right| \le \left\|n^l\right\|_{L^\infty(0,T;L^2(\Omega))} \left\|E^l - E\right\|_{L^2(0,T;L^4(\Omega))} \|\psi\|_{L^2(0,T;L^4(\Omega))}.$$

Since Ω is bounded, we deduce from eq. (27) and (31) that

$$\int_0^T \int_\Omega n^l \left(E^l - E\right)\psi\, dxdt \to 0 \quad (l \to +\infty).$$

On the other hand, let us note that $E\psi \in L^1(0,T;L^2)$. In fact

$$\|E\psi\|_{L^1(0,T;L^2)} \le \|E\|_{L^2(0,T;L^4)} \|\psi\|_{L^2(0,T;L^4)} < \infty.$$

Therefore we deduce from eq. (27) that

$$\int_0^T \int_\Omega (n^l - n)E\psi\, dxdt \to 0 \quad (l \to +\infty).$$

Thus $n^l E^l \to nE$ in $L^2(0,T;H^{-1})$. So $z = nE$.

Third, let ψ be some test function in $L^2(0,T;H^1),\ \operatorname{supp}\psi \subset \Omega \subset \mathbb{R}^3$.

$$\int_0^T \int_{\mathbb{R}^3} \left(f(|E^l|^2)E^l - f(|E|^2)E \right) \psi dxdt$$
$$= \int_0^T \int_\Omega f(|E^l|^2)\left(E^l - E\right)\psi dxdt + \int_0^T \int_\Omega \left(f(|E^l|^2) - f(|E|^2) \right) E\psi dxdt$$

On one hand

$$\left| \int_0^T \int_\Omega f(|E^l|^2)\left(E^l - E\right)\psi dxdt \right| \le \int_0^T \int_\Omega M |E^l|^{2\gamma} \left|E^l - E\right| \left|\psi\right| dxdt$$
$$\le M \left\|E^l\right\|_{L^\infty(0,T;L^{4\gamma}(\Omega))}^{2\gamma} \left\|E^l - E\right\|_{L^2(0,T;L^4(\Omega))} \left\|\psi\right\|_{L^2(0,T;L^4(\Omega))}$$
$$\le C \left\|E^l\right\|_{L^\infty(0,T;H^1(\Omega))}^{2\gamma} \left\|E^l - E\right\|_{L^2(0,T;L^4(\Omega))} \left\|\psi\right\|_{L^2(0,T;L^4(\Omega))}$$

Since Ω is bounded, we deduce from eq. (26) and (31) that

$$\int_0^T \int_\Omega f(|E^l|^2)\left(E^l - E\right)\psi dxdt \to 0 \quad \left(l \to +\infty\right).$$

On the other hand

$$\left| \int_0^T \int_{\mathbb{R}^3} \left(f(|E^l|^2) - f(|E|^2) \right) E\psi dxdt \right| \le \int_0^T \int_\Omega L \left| |E^l|^2 - |E|^2 \right| \left|E\psi\right| dxdt$$
$$\le L \left(\left\|E^l\right\|_{L^\infty(0,T;L^4(\Omega))} + \left\|E\right\|_{L^\infty(0,T;L^4(\Omega))} \right) \left\|E^l - E\right\|_{L^2(0,T;L^4(\Omega))} \left\|E\right\|_{L^\infty(0,T;L^4(\Omega))} \left\|\psi\right\|_{L^2(0,T;L^4(\Omega))}$$

Since Ω is bounded, we deduce from eq. (26) and (31) that

$$\int_0^T \int_{\mathbb{R}^3} \left(f(|E^l|^2) - f(|E|^2) \right) E\psi dxdt \to 0 \quad (l \to +\infty).$$

Thus $f(|E^l|^2)E^l \to f(|E|^2)E$ in $L^2(0,T;H^{-1})$.

Hence taking $l \to \infty$ from eq. (20)-(23), by using the density of ω_j in $H_0^2(\Omega)$ we get the existence of local generalized solution for the periodic initial value problem (4)-(7). letting $D \to \infty$, the existence of local solution for the initial value problem (4)-(7) can be obtain. By the continuation extension principle and a prior estimates, we can get the existence of global generalized solution for problem (4)-(7).

We thus complete the proof of Theorem 2. Hence one can get Theorem 1.

Conclusion

This paper considers the existence of the generalized solution to the Cauchy problem for a generalized Zakharov equation in three dimensions by a priori integral estimates and Galerkin method, one has the existence of the global generalized solution to the problem.

Discussion

One can regard (1)-(2) as the Langmuir turbulence parameterized by $\Gamma\ (0<\Gamma<1)$ and study the asymptotic behavior of the systems (1)-(2) when Γ goes to zero.

Acknowledgments

The author would like to thank the support of National Natural Science Foundation of China (Grant Nos. 11501232) and Research Foundation of Education Bureau of Hunan Province (Grant No. 15B185).

References

1. L.G. Garcia, F. Haas, L.P.L. de Oliveira, J. Goedert, "Modified Zakharov equations for plasmas with a quantum correction", *Phys. Plasmas*, vol. 12, 012302, 2005.
2. F. Haas, P.K. Shukla, "Quantum and classical dynamics of Langmuir wave packets", P*hys. Rev. E*, vol. 79, 066402, 2009.
3. A.P. Misra, D. Ghosh, A.R. Chowdhury, "A novel hyperchaos in the quantum Zakharov system for plasmas", *Phys. Lett. A*, vol. 372, pp. 1469-1476 , 2008.
4. A.P. Misra, S. Banerjee, F. Haas, P.K. Shukla, L.P.G. Assis, "Temporal dynamics in the one-dimensional quantum Zakharov equations for plasmas", *Phys. Plasmas*, vol. 17, 032307, 2010.
5. S. You, "Existence of Global Generalized Solutions for Quantum Zakharov Equation in Dimension Three", *Journal of Interdisciplinary Mathematics*, vol. 18, no. 3, pp. 269-281, 2015.
6. S. You, B. Guo, X. Ning, "Initial Boundary Value Problem for Modified Zakharov Equations", *Acta Mathematica Scientia*, vol. 32B, no. 4, pp. 1455-1466, 2012.
7. S. You, B. Guo, X. Ning, "Equations of Langmuir Turbulence and Zakharov Equations：Smoothness and Approximation", *Appl. Math. Mech*, vol. 33, no. 8, pp. 1079-1092, 2012.
8. J.L. Lions, "Quelques methods de resolution des problemes aux limites non lineaires", Dunod Gauthier Villard, Paris, 12 and 53, 1969.
9. G. Simpson, C. Sulem, P.L. Sulem, "Arrest of Langmuir wave collapse by quantum effects", *Phys. Rev. E*, vol. 80, 056405, 2009.

10. R. Morris, A.H. Kara, A. Biswas, “Soliton solution and conservation laws of the Zakharov equation in plasmas with power law nonlinearity”, *Nonlinear Analysis: Modelling and Control*, vol. 18, no. 2, pp. 153-159, 2013.
11. A.H. Bhrawy, M.A. Abdelkawy, A. Biswas, “Cnoidal and snoidal wave solutions to coupled nonlinear wave equations by the extended Jacobi’s elliptic function method”, *Communications in Nonlinear Science and Numerical Simulation*, vol. 18, no. 4, pp. 915-925, 2013.
12. D.S. Agafontsev, V.E. Zakharov, “Intermittency in generalized NLS equation with focusing six-wave interactions”, *Physics Letters A*, vol. 379, no. 40-41, pp. 2586-2590, 2015.