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Global Solution and Stability Analysis of Constant Delay Stochastic Differential Equation

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Abstract

In this thesis, a global solution and stability of zero solution in connection with constant delay stochastic differential equation is researched. Firstly, generalized the theory is established with the adoption of the Lyapunov function, in order to establish the judging criteria for the constant delay stochastic differential equation, namely the uniqueness of a global solution and stability of its zero solution exponent regarding equation (1) when $y(t)=x(t-T)$. Then, equation coefficients f and g are subject to growth limitations to make generalized conditions specific and acquire conditions which directly depend on equation coefficients. Lastly, the effectiveness of result in this paper is verified through two specific examples.

Key words

Stability Analysis, Constant Delay Stochastic Differential Equation, Global Solution

1. Introduction

Stochastic differential equations of the following forms are researched in this thesis based on existing literature:

$$const|x|^p \le V(x), (x \in \mathbb{R}^n)$$

$$V(t,x,y) \le \zeta(t)\{\Phi K + Q(t)[1+V(x)+V(y)]\}, (t \ge 0, x, y \in \square^n) \quad (1)$$

Where, $f=(f_1,f_2,\ldots,f_n)^T$:$R_+ \times R^n \times R^n \to R^n$ and $g=[g_{ij}]_{n\times m}$:$R_+ \times R^n \times R^n \to R^{n\times m}$ are Borel measurable function; $w(t)$ is m-dimension standard Brownian motion; $x(t)(t \ge 0)$ is the stochastic process of unknown R"; $x(t)$ represents system status at time t. When $y(t)=x(t-T)$, the abovementioned equation is a delay stochastic differential equation. When $y(t)=x(t-\delta(t))$, where $\delta(t)$ is a variation delay, the abovementioned equation will be a variational delay stochastic differential equation.

In this thesis, $t_0=0$ is always the initial time. It is assumed that $x(t,\xi)$ is the solution to the given equation when ξ is the initial value. In case it is defined as $x(t,\xi)$, it will be considered a global solution. The global solution is important because its existence is the precondition for research on the asymptotic nature of the equation, without which it would be impossible to observe its asymptotic nature. Secondly, it is always assumed that coefficients of the equation are such that $f(t,0,0)=g(t,0,0)=0$, which ensures that there will be a trivial solution $x(t,0)$ for equation (1). It is the precondition for research on real solutions to the equation and the stability of its numerical solution and zero solution.

2. Stochastic stability theory

The existence of a global solution is the precondition for research on the asymptotic nature of a solution to equation. To put it intuitively, the existence of a global solution means that the solution to the equation will not blast within any limited time period. Among the existing theories concerning the stochastic differential equation, a number of proven results have been provided regarding the uniqueness of of a global solution to the stochastic differential equation with bounded delay. In order to guarantee that there is only one unique global solution for any initial value or equation, the equation coefficient will, in most cases, either satisfy the liner growth condition and the local Lipschitz condition [9] or the non-Lipchitz condition and the liner growth condition [8]. In this thesis, the linear growth condition for the existence of a global solution is replaced by conditions that are more common [10].

Stability has been extensively researched as the central subject of either a deterministic or stochastic dynamical system theory. In 1892, Lyapunov A.M.[4], a Russian mathematician, introduced the concept of stability in a power system. In brief, stability theory is applicable to such problems: ultimate state of $x(t,\xi)$, solution to a differential equation, when $t \to \infty$; dependence relation between such ultimate state and the initial value ζ. The answer to such questions is related to the long-term behavior of the development system described in the equation, to which researchers of different fields all pay close attention. No matter in theories concerning an ordinary differencing equation or theories concerning a stochastic differential equation, stability is always the most heeded central subject, and related literature constitute a large portion of theories concerning differential equations.

The development of stochastic stability theory is based on stability theory of a deterministic system. The deterministic system described in ODE is:

$$\dot{x}(t) = f\left(t, x(t)\right), t \geq t_0 \tag{2}$$

Initial value $x(t_0)=x_0 \in R^n$, the stability theory was first considered by Hahn [5] and Lakshmikantham, et al. [6]. If an explicit solution to equation (1) can be acquired when $f(t,0)=0$, it would be easy to judge if its zero solution is stable. However, in most cases, equations in the form of equation (1) are not likely to have an explicit solution. Lyapunov proposed a method in 1892 [4] during his research on the stability of deterministic systems. With adoption of such a method, there is no need to solve the equation. It is a method to directly determine the stability by adopting the said Lyapunov function $V(t,x)$ and symbols of derivative $V(t,x)$ with a disturbed solution. Such a method is called the Lyapunov direct method or the Lyapunov second method.

Application of Lyapunov stability theory to a stochastic power system will cause new problems. Firstly, there will be a reasonable definition for stochastic stability; secondly, how should one select Lyapunov function $V(t,x)$ and should one generalize stability conditions for equations in the form of $V(t,x)<0$? Stochastic stability will contain at least three different types: probability-based stability, moment stability and orbital stability, for which there are detailed definitions in books on stochastic differential equation [1]. In 1965, Bucy [7] proposed that the stochastic Lyapunov function will be equipped with the abovementioned properties and proposed sufficient conditions of probability-based stability and moment stability. Has'minskii studied orbital stability of a linear stochastic differential equation in 1967 [8]. There is a wide range of research on stability in theories concerning stochastic differential equations and many mathematicians have done much related work. Representational literature includes [11, 13].

It this thesis, the Lyapunov function is adopted to establish a generalized principal, and it proposes the judging criteria for the constant delay stochastic differential equation; namely the uniqueness of a global solution and stability of its zero solution exponent regarding equation (1) when $y(t)=x(t-T)$. Then, the equation coefficients f and g are subject to growth limitation based on further results, which specifies the generalized conditions and acquires conditions to depend directly on equation coefficients. Lastly, the effectiveness of results in this paper is verified through two specific examples.

3. Uniqueness and stability of global solution

Firstly, in terms of function $v(x)$ which satisfies certain conditions, the judging criteria for the uniqueness of a global solution to equation (1) is established by proposing growth limiting conditions of $v(t,x,y)$.

According to theorem 1, there are positive constants K and P, non-negative decreasing function, non-negative increasing function $Q(t)$ and function $V(x)\in C^2(R^n)$, and that:

$$const|x|^p \le V(x),(x \in \square^n) \tag{3}$$

$$V(t,x,y) \le \zeta(t)\{\Phi K + Q(t)[1+V(x)+V(y)]\},(t \ge 0, x, y \in \square^n) \tag{4}$$

where Φ_K is based on (1). Then $\forall\zeta\in C$ and equation (3) has unique global solution $x(t,\zeta)$.

For verification, set $\forall\zeta\in C$. As equation coefficients f and g satisfy the assumed theorem 1, it can be verified by adopting a standard truncation function technique [18] and a theorem similar to theorem 2 in [14] where equation (3) has a unique maximum local solution $x(t)=x(t,\zeta)(-\tau\le t<\sigma)$, where σ is the blasting time.

Set k_0 is a sufficiently large positive number which make $||\xi||k_0$. For any $k\le N$ and $k\le N$, and establish that for the time-stopping sequence, σ_k is obviously monotonically increasing for k when $k\ge k_0$. In case $k\ge k_0$, and that $\{-\tau\le t<\sigma:|x(t)\ge k|\}=\Phi$, then $x(t)$ is bounded within $[-\tau,\sigma)$. This means that $\sigma=\propto$. In this case, it is obvious that $\sigma=\propto$. In case of any $k\ge k_0$, there is $\{-\tau\le t<\sigma:|x(t)\ge k|\}=\Phi$, then it is definite that $\sigma_k\le\sigma$; therefore, there is also $\sigma_\propto\le\sigma$. It can be ascertained that to verify $\sigma=\propto$, a, s, it is only necessary to verify $\forall t\ge 0$, when $k\to\propto$, $P(\sigma_k\le t)\to 0$.

$$\begin{aligned} kP(\sigma_k \le t) &\le P(\sigma_k \le t)V(x(\sigma_k)) \\ &= E\left[I_{\{\sigma_k \le t\}}V(x(\sigma_k))\right] \le h_k(t) \end{aligned} \tag{5}$$

According to condition (4) and lemma 1:

$$
\begin{aligned}
h_k(t) &= EV(x(0)) + E\int_0^{t_k} LV(x(\mathrm{s}))ds \\
&\le const + E\int_0^{t_k} \zeta(s)\left\{\Phi K(s, x(s), y(s)) + Q(s)\left[1 + V(x(s)) + V(y(s))\right]\right\} ds \\
&\le const + constQ(t)E\int_0^{t_k}\left[1 + V(x(s)) + V(y(s))\right]ds \\
&\le const + constQ(t)t + constQ(t)E\int_0^{t_k}\left[V(x(s)) + V(x(s-\tau))\right]ds \\
&\le const(1 + tQ(t)) + constQ(t)E\int_0^{t_k} V(x(s))ds + constQ(t)E\int_0^{t_k} V(x(s))ds \\
&\le M(t) + N(t)\int_0^t h_k(s)ds
\end{aligned}
\tag{6}
$$

where $M(t)$ and $N(t)$ are non-negative increasing function and its expression is not required.

According to lemma 1:

$$h_k(t) \le M(t)e^{tN(t)} \tag{7}$$

Thus, $kP(\sigma_k \le t) \le M(t)e^{tN(t)}$; therefore, when $k \to \propto$:

$$P(\sigma_k \le t) \le k^{-1}M(t)e^{tN(t)} \to 0 \tag{8}$$

Then it is verified.

The most common method to determine uniqueness of a global solution to equation (3) is the following simple method. The special Lyapimov function $V(x)$ is adopted, which can be deemed as a deduction of theorem 1.

Based on theorem 2, set positive constants P and K and non-negative decreasing function $\xi(t)$, make $V(x)=|x|^p$ satisfy:

$$V(t, x, y) \le \zeta(t)(\Phi(k) + const), (t \ge 0, x, y \in R^n) \tag{9}$$

where $\Phi(k)$ is based on equation (90). $\forall \xi \in c$ and there is a unique solution $x(t,\xi)$ for equations (1, 2, 3).

Firstly, it is assumed that any $t \ge 0$, and $f(t,0,0)=g(t,0,0)\equiv 0$. Therefore, $x(t,0)\equiv 0$; namely, the zero solution for equation (1) is its equilibrium solution.

In this thesis, exponential stability is the research subject. There are two cases, namely p-moment exponential stability and almost surely orbital exponential stability, which are expressed as the following asymptotic estimation:

$$\limsup_{t \to \propto} \frac{\ln E\left|x(y, \zeta)\right|^p}{t} \le -\gamma \tag{10}$$

$$\lim_{t\to\infty}\sup\frac{\ln E\left|x\left(y,\zeta\right)\right|}{t}\leq -q,\quad a.s. \tag{11}$$

In this section, judging criteria for stability of equation (1) are established with the adoption of the properly selected Lyapunov function $V(x)$ and are based on growth limitations for $V(t,x,y)$. Compared with the foregoing articles and comprehending in an intuitive manner, stability has a stronger limitation than boundedness for growth tendency of the equation's solution. In terms of stability, the solution will tend asymptotically to an equilibrium solution, which creates a growth limitation for $V(t,x,y)$ stronger than control conditions in foregoing articles, and stability is acquired when the uniqueness of a global solution to the equation is ensured.

Firstly, the following generalized theorem is established based on the semi martingale convergence theorem.

In terms of theorem 3, set positive constant P, and this makes function $V(x)=|x|^p$ satisfy:

$$V\left(t,x,y\right)\leq \Phi\left(k\right)+aV\left(x\right),\left(t\geq 0,x,y\in R^n\right)$$

(12)

where $\Phi(k)$ is based on equation (9). Then there is $\gamma=a\wedge\frac{\ln K}{\tau}$; for any initial value $\xi\in c$, and there will be a unique global solution $x(t,\xi)$ for equation (1), which satisfies equations (6) and (7).

If condition (8) in theorem 2 is satisfied based on $V(x)=|x|^p$ and condition (7), it then can be deduced by adopting theorem 2 that for any initial value $\xi\in c$, there will be a unique global solution $x(t,\xi)$ for equation (1).

The initial value is set as $\xi\in c$, and $x(t)= x(t,\xi)$, $h(t)=e^{n}V(x(t))$

$$h\left(t\right)=h\left(0\right)+\int\ e^{rt}[V\left(x\left(t\right)\right)+\gamma\ \mathrm{V}\left(x\left(s\right)\right)]\,\mathrm{d}s+M\ \left(t\right) \tag{13}$$

$$M\ \left(t\right)=\int_0^t e^{rt}V_x\left(x\left(s\right)\right)g\left(s,x\left(s\right),\mathrm{y}\left(s\right)\right)dw\left(s\right)+\gamma\ \mathrm{V}\left(x\left(s\right)\right)]\,\mathrm{d}s \tag{14}$$

$$\int_0^t e^{rt}[V_x\left(x\left(s\right)\right)+\gamma V\left(x\left(s\right)\right)]ds\leq\int_0^t e^{rt}[\Phi k\left(s,x\left(s\right),y\left(s\right)\right)-aV\left(x\left(s\right)\right)+\gamma V\left(x\left(s\right)\right)]ds\leq const$$

(15)

At the last step, lemma 3 and $\gamma=a\wedge\frac{\ln K}{\tau}$ are adopted. Therefore, $h(t)\leq const+M(t)$. Thus, it can be obtained according to semi-martingale convergence theorem that:

$$\limsup_{t\to\infty} Eh(t) < \infty, \quad \limsup_{t\to\infty} h(t) < \infty \quad a.s. \tag{16}$$

It can be respectively deduced that equations (7) and (8) are true.

4. Further results

Next, the theorems in last section are adopted to obtain a series of more applicable judging theorems of which conditions are more specific. Further specification is realized with the selection of specific function $V(x)$, to make it directly related to equation coefficients f and g, which are more applicable in practical cases.

Firstly, the following two alternative assumed conditions are provided for equation coefficients f and g:

Assumption 1: (unilateral linear growth), set σ as a positive constant, $\bar{\sigma}$ and λ that $\bar{\lambda}$ is a non-negative constant. For any $(t,x,y) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$,

$$2x^T f(t,x,y) \le -\sigma|x|^2 + \bar{\sigma}|y|^2 \tag{17}$$

$$|g(t,x,y)|^2 \le -\lambda|x|^2 + \bar{\lambda}|y|^2 \tag{18}$$

Assumption 2: (multinomial growth), set σ0 and σ as positive constants, $\sigma_l, \bar{\sigma}_l, \lambda_l$. And that $\bar{\lambda}_l$ is a non-negative constant. $0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_L \le \alpha$. For any $(t,x,y) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$

$$2x^T f(t,x,y) \le -\sigma_0|x|^2 + \bar{\sigma}|y|^{2+\alpha} + \sum_{l=1}^{L}\left(\sigma_l|x|^{\alpha l+2} + \bar{\sigma} l|y|^{\alpha l+2}\right) \tag{19}$$

$$|g(t,x,y)|^2 \le \sum_{l=1}^{L}\left(\lambda_l|x|^{\alpha l+2} + \bar{\lambda}_l|y|^{\alpha l+2}\right) \tag{20}$$

The following result can be obtained with adoption of theorem 3:

Theorem 4, assuming that 1 is true, then:

$$\sigma > \iota + \bar{\sigma} + \bar{\lambda} \tag{21}$$

For any initial value $\xi \in c$, equation (1) will have a unique global solution $x(t,\xi)$, which satisfies equations (8) and (9). Where y is the only positive root:

$$\sigma\text{-}\lambda\text{-}\gamma > (\bar{\sigma} + \bar{\lambda})e^{\gamma\tau} \tag{22}$$

of equation (22) Set $V(x)=|x|^2$, and based on equation (15):

$$V(t,x,y) = 2x^T f(t,x,y) + |g(t,x,y)|^2 \tag{23}$$

Substitute conditions (11) and (12) into the equation above:

$$\begin{aligned} V(t,x,y) &\le -\sigma|x|^2+\overline{\sigma}|y|^2+\lambda|x|^2+\overline{\lambda}|y|^2 \\ &= -(\sigma-\lambda)|x|^2+(\overline{\sigma}+\overline{\lambda})|y|^2 \end{aligned} \tag{24}$$

Set $k=e^{rt}$, and it can be obtained based on the equation above that:

$$\begin{aligned} V(t,x,y) &\le U(y)-KU(x)-\left[\sigma-\lambda-K\left(\overline{\sigma}+\overline{\lambda}\right)\right]|x|^2 \,\square\, \Phi_K-\alpha|x|^2 \\ &= -(\sigma-\lambda)|x|^2+(\overline{\sigma}+\overline{\lambda})|y|^2 \end{aligned} \tag{25}$$

Where $U(x)=\left(\overline{\sigma}+\overline{\lambda}\right)|x|^2$, and Φ_k is based on equation (12), $\alpha=\sigma\text{-}\lambda\text{-}K\left(\overline{\sigma}+\overline{\lambda}\right)$.

Based on theorem 3 it can be ascertained that equations (8) and (9) are true. Therefore, the conclusion can be verified as true. Similarly, the following theorem can be obtained.

Theorem 4: σ is a positive constant, σ_t, $\overline{\sigma_t}$, λ_t. And $\overline{\lambda_t}$ is non-negative constant which satisfies:

$$2x^T f(t,x,0) \le -\sigma|x|^2 \tag{26}$$

$$\left|f(t,x,y)-f(t,x,0)\right| \le -\sigma|y|^2 \tag{27}$$

$$\left|g(t,x,y)\right|^2 \le \lambda|x|^2+\lambda|y|^2 \tag{28}$$

For any $x,y\in\square^n$ and that $t\ge 0$. If:

$$\sigma > 2\sqrt{\overline{\sigma}}+\lambda+\overline{\lambda} \tag{29}$$

For any initial value $\zeta\in C_{p_0}^b\left([-\tau,0];\square^n\right)$, equation (1) will have a unique global solution $x(t,\xi)$ which satisfies equations (8) and (9). Where y is the only positive root of the following equation:

$$\sigma\text{-}\overline{\sigma}\text{-}\lambda\text{-}\gamma > (\overline{\sigma}+\overline{\lambda})\; e^{\gamma\tau} \tag{30}$$

Theorem 5 set assumption (18) is true, and:

$$\sigma\wedge\frac{\sigma_0}{\rho} > \sum_{l=1}^{L}\left(\sigma_l+\lambda_l+\overline{\sigma_l}+\overline{\lambda_l}\right) \tag{31}$$

For r any initial value $\xi\in c$, equation (1) will have a unique global solution $x(t,\xi)$, which satisfies equations (8) and (9).

Verification. Set $V(x)=|x|^2$, and according to equation (15):

$$V(t,x,y)=2x^{T}f(t,x,y)+\left|g(t,x,0)\right|^{2} \tag{32}$$

Substitute conditions (13) and (14) into the above equation:

$$V(t,x,y)\le -\sigma_0|x|^2-\sigma|x|^{\alpha+2}+\sum_{l=1}^{L}\left((\sigma l+\lambda l)|x|^{\alpha l+2}+(\overline{\sigma}l+\overline{\lambda}l)|y|^{\alpha l+2}\right) \tag{33}$$

Set $K_0=\left[\sigma\wedge\frac{\sigma_0}{\rho}\right]-\sum_{l=1}^{L}(\sigma_l+\lambda_l)\Big/\sum_{l=1}^{L}(\overline{\sigma}_l+\overline{\lambda}_l)$; in case $\sum_{l=1}^{L}(\overline{\sigma}_l+\overline{\lambda}_l)=0$, set $K_0=\infty$.

Therefore, constant K satisfies $1<k\le k_0$:

$$V(t,x,y)\le -\sigma_0|x|^2-\sigma|x|^{\alpha+2}+\sum_{l=1}^{L}\left((\sigma l+\lambda l)|x|^{\alpha l+2}+(\overline{\sigma}l+\overline{\lambda}l)|y|^{\alpha l+2}\right)+U(y)+KU(x) \tag{34}$$

Where $U(x)=\sum_{l=1}^{L}(\sigma_l+\lambda_l)|x|^{\alpha l+2}$.

As $k\le k_0$, $\sigma>\sum_{l=1}^{L}\left(\sigma_l+\lambda_l+K(\overline{\sigma}_l+\overline{\lambda}_l)\right)$. Apply the above equation to theorem 3:

$$\begin{aligned}V(t,x,y)&\le -\sigma|x|^{\alpha+2}+\sum_{l=1}^{L}\left((\sigma_l+\lambda_l)|x|^{\alpha l+2}+(\overline{\sigma}_l+\overline{\lambda}_l)|y|^{\alpha l+2}\right)\\&\le \rho\sum_{l=1}^{L}\left(\sigma_l+\lambda_l+K(\overline{\sigma}_l+\overline{\lambda}_l)\right)|x|^2\end{aligned} \tag{35}$$

Where ρ is based on equation (17) and substitute it into equation (15) to obtain the equation:

$$V(t,x,y)\le \Phi_K-\alpha V(x(s)) \tag{36}$$

Where Φ_k is based on equation (12), $a=\sigma_0-\rho\sum_{l=1}^{L}\left(\sigma_l+\lambda_l+K(\overline{\sigma}_l+\overline{\lambda}_l)\right)$. Set $\gamma=a\wedge\frac{\ln K}{\tau}$, it can be known that equations (8) and (9) are true based on theorem 3.

Next, we will try to find the most suitable $K\in(1,K_0]$ to make $\gamma=a\wedge\frac{\ln K}{\tau}$ realize the maximum value. It is obvious that the value of a decreases as k increases. Set $a=\frac{\ln K}{\tau}$, and assume that K1 is the only positive root of the following equation:

$$\frac{\ln K}{\tau}=\sigma_0-\rho\sum_{l=1}^{L}\left(\sigma_l+\lambda_l+K(\overline{\sigma}_l+\overline{\lambda}_l)\right) \tag{37}$$

It is obvious that $k_1>1$. In case $k_1\le k_0$, set $k\le k_1$ and $\gamma=a=\frac{\ln K_1}{\tau}$.

In case $k_l>k_0$, set $k=k_l$ and $a=\sigma_0-\rho\sum_{l=1}^{L}\left(\sigma_l+\lambda_l+K\left(\overline{\sigma_l}+\overline{\lambda_l}\right)\right)$. It can be known from $k_l>k_0$ and definition of K_1 that:

$$\frac{\ln K}{\tau}=\frac{\ln K_1}{\tau}=\sigma_0-\rho\sum_{l=1}^{L}\left(\sigma_l+\lambda_l+K\left(\overline{\sigma_l}+\overline{\lambda_l}\right)\right)<\alpha$$

$$\gamma=a\wedge\frac{\ln K_0}{\tau}=\frac{\ln K_0}{\tau} \tag{38}$$

Therefore, the conclusion to be verified is true.

5. Exemplification and analysis

In this section, the effectiveness of the judging theorem obtained in the foregoing section will be illustrated through two specific examples.

Example 1: in terms of the following two dimensional stochastic differential equation:

$$dx_1(t)=\left[-5x_1(t)-x_1^3(t)+\sqrt{2}x_2(t-1)\right]dt+x_1(t)dw(t)$$

$$dx_2(t)=\left[-5x_1(t)-5x_2(t)\text{-}x_2^3(t)+x_1(t-1)\right]dt-x_2(t-1)dw(t) \tag{39}$$

Where $x=(x_1,x_2)^T\in\mathbb{R}^2$ and $w(t)$ is one-dimensional Brown motion.

The stability of equation (39) is tested with the adoption of theorem 1. Defining $y(t)=x(t-1)$ arrives at:

$$f_1(t,x,y)=-5x_1-x_1^3+\sqrt{2}y_2$$

$$f_2(t,x,y)=5x_1-5x_2-x_2^3+y_1$$

$$g_1(t,x,y)=x_1$$

$$g_1(t,x,y)=x_1$$

$$f(t,x,y)=\left(f_1(t,x,y),f_2(t,x,y)\right)^T$$

$$g(t,x,y)=\left(g_1(t,x,y),g_2(t,x,y)\right)^T \tag{40}$$

Equation (3) then can be rewritten in standard form as

$dx(t)=f\left(t,x(t),y(t)\right)dt+g\left(t,x(t),y(t)\right)dw(t)$.

It can be obtained with adoption of Young in equation that:

$$2x^T f(t,x,y) \le -4|x|^2 + |y|^2$$

$$|g(t,x,y)|^2 = |x|^2 + |y|^2 \tag{41}$$

Therefore, set $\tau=1, \sigma=4, \lambda=1, \bar{\lambda}=1$, to make assumption (1) is satisfied. $\sigma=4 > \lambda + \bar{\sigma} + \bar{\lambda}=3$, it can be learned from theorem 1 that, equation (39) has a stable zero solution exponent no matter in terms of mean square or orbit. Namely, equation 39 satisfies equations (8) and (9). Where γ is the unique positive root of the following equation:

$$3\text{-}\gamma=2e^{\gamma} \tag{42}$$

Based on approximate calculation, $\gamma\approx0.3001$

Example 2: following is the test of another stochastic delay differential equation:

$$dx_1(t) = \left[-x_1(t) - 3x_1^3(t) + x_2{}^2(t-1)\right]dt + |x_2(t)|^{\frac{3}{2}} dw(t)$$

$$dx_2(t) = \left[-x_2(t) - 3x_2^3(t) + x_1{}^2(t)\right]dt + |x_1(t-1)|^{\frac{3}{2}} dw(t) \tag{43}$$

Where $x=(x_1,x_2)^T \in \mathbb{R}^2$ and $w(t)$ is one-dimensional Brown motion.

Stability of equation (43) will be verified with adoption of equation (43). Define $y(t)=x(t-1)$, then:

$$f_1(t,x,y) = x_1 - 3x_1^3 + y_2{}^2$$

$$f_2(t,x,y) = x_1{}^2 - x_2 - 3x_2^3$$

$$g_1(t,x,y) = |x_2|^{\frac{3}{2}}$$

$$g_2(t,x,y) = |y_1|^{\frac{3}{2}}$$

$$f(t,x,y) = \left(f_1(t,x,y), f_2(t,x,y)\right)^T$$

$$g(t,x,y) = \left(g_1(t,x,y), g_2(t,x,y)\right)^T \tag{44}$$

Equation (43) then can be rewritten in standard form as $dx(t) = f(t,x(t),y(t))dt + g(t,x(t),y(t))dw(t)$. It can be obtained with adoption of Young in equation that:

$$2x^T f(t,x,y) \le -2|x|^2 - 6|x|^4 + 2|x|^3 + \frac{4}{3}|y|^3$$

$$|g(t,x,y)|^2 \le |x|^3 + |y|^3 \quad (46)$$

Therefore, set $\tau=1, \sigma_0=2, \sigma=6, \sigma_1=2, \bar{\sigma}=4/3, \lambda_1=1, \bar{\lambda}=1$, to satisfy assumption (18). It is obvious that $\sigma \wedge \frac{\sigma_0}{\rho}=6 \wedge 8=6 > \sigma_1+\bar{\sigma}+\lambda_1+\bar{\lambda}=5\frac{1}{3}$, where p = 1/4 according to equation (17). It can be known from theorem 3 that, equation (43) has stable zero solution exponent no matter in terms of mean square or orbit. Namely, equation 43 satisfies equations (8) and (9). Set:

$$K_0 = \frac{\sigma \wedge \frac{\sigma_0}{\rho} - (\sigma_l - \lambda_l)}{\bar{\sigma}_l + \bar{\lambda}_l} = \frac{9}{7} \quad (47)$$

K1 is the unique positive root of the following equation:

$$\ln K = 2 - \frac{1}{4}\left(3 + \frac{7}{3}K\right) \quad (48)$$

Based on approximate calculation, $K_1 \approx 1.4757$. As $K_1 \approx 1.4757 \ge K_0 = \frac{9}{7} \approx 1.2857$, it can be known from theorem 3 that:

$$\gamma = \frac{\ln K_1}{\tau} \approx 0.2513 \quad (49)$$

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