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Identification of Monoparametric Families of a Remarkable Complex Bol Algebras Class, Generated by the Symmetric Space $g_2/so(4)$

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Abstract

Our paper focuses on identification and classification of remarkable Bol algebras class generated by exceptional compact symmetric pair $g_2/so(4)$ and characterized by the identity $(x.y).(z.u)=0$. This study is based on a matrix realization of g_2 as an algebra of anti-symmetric matrixes of order 8×8. These matrixes verify a system of invariant identities imposed by isomorphism that exists between g_2 and the octonions derivations algebra. The aim of the research and its results are a part of a common classic investigation approach that specifically targets classification of mathematical studied objects . In the treated case of couple $g_2/so(4)$, our work focuses on *Bol* Algebras that represent the linear infinitesimal analog of local analytic (left) *Bol* loops generated by the homogeneous space $G_2/SO(4)$ that is symmetric, compact and of rank 2. The paper contains a complete list of the monoparametric mentioned algebras families that appear as complex solutions of a constructed matrix equations system.

Key words

Bol Algebras, *Bol* Loops, *Mal'cev* Algebras, *Moufang* Loops, Binary *Lie* Algebras, Diassociative Loops, *Lie* Triple Systems, Symmetric Pair $g_2/so(4)$, *Octonions*.

1. Introduction

Introduction of "group" concept by *Évariste Galois* was surely a great turning point in science history [2]. *Henri Poincaré* wrote: "The concept of group pre-exists in our minds, at least potentially. It is imposed on us not as a form of our sensitivity, but as a form of our understanding". To statistically estimate the role of "group" in modern sciences, just for example consider the number of papers treating groups in a yearly sample of a certain journal. For journal Advances in Modeling, series A (year 2005), we find 5 publications about groups [4,20-22,24].

Mathematic developments based on group concept had proven efficiency of this fundamental being for treating questions of solvability of algebraic equations by radicals, as well as questions of constructability using rule and compass in geometry. On footsteps of *Galois* researches, works of *Sophus Lie, Felix Klein, Henri Poincaré, Elie Cartan, Wilhelm Killing*, and many others, have revealed the big importance of concept "differentiable group" in several branches of mathematics and mechanics. Particularly, researches of these scientists have shown possibility of adequate local infinitesimal approximation of differentiable manifold having a group structure (*Lie* group), using the corresponding tangent space having a linear algebraic structure (*Lie* algebra). The correspondence between the geometric structures (curvilinear) and the algebraic structure (linear) is locally one-to-one, what allows application of algebraic methods for local resolution of geometric problems. In 1955, *Anatoly Mal'cev* [7] noticed that theory of *Lie* could be generalized on differentiable manifolds equipped with a diassociative loop structure (in this case, any two elements of the manifold generate a sub-group, the tangent space will then be a binary *Lie* algebra). *Mal'cev* has particularly developed a theory similar to *Lie*'s one, for differentiable manifolds of *Moufong* loops (diassociative loops that verify the identity said of *Moufong*: $z.(x.(z.y))=((z.x).z).y$). The obtained algebras in this case are named *Mal'cev* algebras and are characterized by an internal law *(.)*, bilinear, anticommutative, and verifying identity:

$$(x.y).(x.z) = ((x.y).z).x + ((y.z).z).x + ((z.x).x).y.$$

Based on *Lie* and *Mal'cev* theories, as well as on results establishing equivalence between homogeneous spaces and loops [13,14], *L. Sabinin* and *P.Mikheev* have developed a theory about local linear infinitesimal element of the differentiable (left) *Bol* loop (loop that verifies the left identity of *Bol*: x(y(xz))=(x(yx))z). Notice that *Moufang* loop could be defined as a loop verifying at once the left identity of *Bol*, as well as the right identity (((zx)y)x=z((xy)x)), so that the theory of *Sabinin-Mikheev* generalizes the theory of *Mal'cev* and *a fortiori Lie*'s one. We also must notice that theory of *Sabinin-Mikheev* is strongly related to works of *M. Akivis* and his disciples in the area of *Bol three webs* [1].

Thus, *Bol* algebras generalize those of *Lie*, as well as those of *Mal'cev* [5,7]. The *Lie* functor, establishing equivalence between the category of local *Lie* groups and the category of *Lie* algebras, could be generalized, not only in case of local analytic diassociative (resp. *Moufang*) loops, and binary *Lie* (resp. *Mal'cev*) algebras, but this functor has also a generalization in cases of *Bol* local analytic loops and *Bol* algebras [8,15-19]. These latter algebras representing the linear infinitesimal analog of local analytic *Bol* loops, are characterized, contrary to classical cases of *Lie* and *Mal'cev*, by two internal composition laws; the first one is bilinear, anti-commutative and expressing the "*deviation measurement*" of the composition law in the loop from the commutativity; the other law is ternary, trilinear, anti-symmetric according to the two first arguments, and it expresses the " *deviation* measurement" loop law from the associativity.

Symmetric spaces being homogeneous, generalizing the physic-geometric concept of symmetry, their linear infinitesimal analogs, *Bol* algebras, are surely important mathematics models for mathematics and physics sciences [13,14].

We have already realized the classification of *Bol* algebras associated to compact symmetric spaces of rank 1 [26-28]. Let's mention, particularly, that in the case of exterior cubic automorphism of $so(8)$ (triality automorphism), relatively to the binary law of *Bol* algebra generated by the symmetric pair $so(8)/so(7)$, we obtained the 7-dimensional simple algebra of *Mal'cev* that isn't of *Lie* [28].

Our paper is devoted to classification of monoparametric families of *Bol* algebras of remarkable class associated to the symmetric couple $g_2/so(4)$ and characterized by identity $(x.y).(u.v)=0$. Addressing the problem is justified by the following factors:

- Examples of algebras that are of *Bol* without being of *Mal'cev* (particularly of *Lie*) are too rare according to our knowledge. Our classification offers a large variety of such algebras.
- *Bol* algebras are characterized by 2 internal composition laws, one being binary and the other one is ternary. These two laws are weakly linked, and it appears that elaboration of a structural theory achieving these algebras is far from being easy [see for example 9-12]. Consequently, inductive accumulation of partial and concrete results concerning these algebras will be justifiable and inevitable during researches going on elaboration of a structural theory.
- The considered problem is a step towards classification of *Bol* algebras associated to symmetric spaces (particularly to spaces of rank 2). Results must then offer additional information on mathematical objects already introduced and adopted by big classics of sciences in different mathematical areas. Particularly, the mentioned classification must offer linear mathematic structures modeling locally a large variety of "geometric symmetries".
- Frequent use of octonions and of hyper-complex systems to model physical and mathematical problems.

2. *Lie* triple systems and *Bol* algebras

Let T be a n-dimensional vector space, defined over a commutative field $\mathcal{K}$. We design by θ the trivial vector of T.

Definition 1. The vector space T is said a *Lie* triple system, if there's a ternary internal composition law defined on T:

$$\begin{array}{llll} [-,-,-]: & \boldsymbol{T}\times\boldsymbol{T}\times\boldsymbol{T} & \to & \boldsymbol{T} \\ & (\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) & \to & [\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}], \end{array}$$

satisfying the following axioms:

1. the [-,-,-] law is linear for all its arguments,
2. $(\forall\, x,y,z \in T)\ \ [x,y,z] = -[y,x,z]$ and $[x,x,y] = \theta$,
3. $(\forall\, x,y,z \in T)\ \ [x,y,z] + [z,x,y] + [y,z,x] = \theta$,
4. $(\forall\, x,y,z,u,v \in T)\ \ [x,y,[z,u,v]] = [[x,y,z],u,v] + [z,[x,y,u],v] + [z,u,[x,y,v]]$.

(If $char\mathcal{K} \neq 2$, the two conditions in axiom 2 are equivalent).

Definition 2. The *Lie* triple system T is said (left) *Bol* algebra, if there's a second binary internal composition law defined on T:

$$\begin{array}{lll} (.): & T\times T & \to T \\ & (x,y) & \to x.y, \end{array}$$

satisfying the following additional axioms:

5. $(\forall\, x,y \in T)\ \ x.y = -y.x$ and $x.x = \theta$,
6. $(\forall\, a,b,x,y \in T)\ \ [a,b,x.y] = [a,b,x].y + x.[a,b,y] + [x,y,a.b] + (a.b).(x.y)$.

(If $char\mathcal{K} \neq 2$, the two conditions in axiom 5 are equivalent).

Let $\Gamma = L \oplus E$ a *Lie* algebra represented as a direct sum of vector subspaces L and E, where L is a subalgebra of Γ and E is a vector subspace.

Definition 3. The pair *(L,E)* is said symmetric if:

1: $[L,E] \subseteq E$, 2. $[E,E] \subseteq L$.

Proposition 1. [6,25] The pair *(L,E)* is symmetric if and only if it exists an automorphism $\sigma: \Gamma \to \Gamma$ such that : $\sigma(x) = \begin{cases} x & \text{if } x \epsilon L \\ -x & \text{if } x \epsilon E \end{cases}$

For the proof, see for example [6].

Proposition 2. If *(L,E)* is a symmetric pair, the ternary law introduced by relation:

$$[x, y, z] \overset{\text{def}}{=} [[x, y], z], (x, y, z \in E),$$

equips E with a *Lie* triple system structure (for the proof, see for example [6]).

Let T be a *Lie* triple system. By $End_v(T)$ we designate the space of vector endomorphisms of T.

Definition 4. The linear application $\Delta: T \to T$, $(\Delta \in End_v(T))$ is said derivation of T, if:

$(\forall x, y, z \in T) \quad \Delta[x, y, z] = [\Delta x, y, z] + [x, \Delta y, z] + [x, y, \Delta z].$

N.B. The vector space defined over field $\mathcal{K}$ and generated by derivations of T is designed by $\mathcal{D}(T)$.

Proposition 3. [6,15,25] Equipped with the commutation law

$$[\Delta_1, \Delta_2] = \Delta_1\Delta_2 - \Delta_2\Delta_1, (\Delta_1, \Delta_2 \in \mathcal{D}(T)),$$

the vector space $\mathcal{D}(T)$ is a *Lie* algebra.

For the proof, see for example [6].

Let a and b be two elements of a triple system T. Let's introduce the application $D_{a,b}$ as:

$$\begin{aligned} D_{a,b}: \quad & T \to T \\ & x \to D_{a,b}(x) \overset{\text{def}}{=} [a, b, x] \end{aligned}$$

Assume, by definition $\mathcal{D}_0(T) \overset{\text{def}}{=} Vect_K \{D_{a,b}: \ a, b \in T\}$.

Proposition 4.

1. $(\forall a, b \in T) \quad D_{a,b} \in \mathcal{D}(T)$.

2. $\mathcal{D}_0(T)$ is a subalgebra of the *Lie* algebra $\mathcal{D}(T)$.

Proof.

1. $(\forall x, y, z \in T)\ (\forall \alpha, \beta \in K\ )\quad D_{a,b}(\alpha x + \beta y) = [a, b, \alpha x + \beta y] = \alpha[a, b, x] + \beta[a, b, y]$

   $= \alpha D_{a,b}(x) + \beta D_{a,b}(y),$

   so $D_{a,b}$ is linear. On the other hand,

   $$D_{a,b}[x, y, z] = [a, b, [x, y, z]]$$

   $$= [[a, b, x], y, z] + [x, [a, b, y], z] + [x, y, [a, b, z]] = [D_{a,b}(x), y, z] + [x, D_{a,b}(y), z] + [z, y, D_{a,b}(z)]$$

   ,

   hence $D_{a,b} \in \mathcal{D}(T)$.

2. It's sufficient to prove the stability of $\mathcal{D}_0(T)$ for the commutation. According to identity 4 (definition 1),

   $(\forall x, y, u, v \in T)\ [D_{x,y}, D_{u,v}] = D_{x,y}D_{u,v} - D_{u,v}D_{x,y} = D_{[x,y,u],v} + D_{u,[x,y,v]} \in \mathcal{D}_0(T).$

Let's now introduce the vector space:

$$[T,T]\oplus T = \mathcal{D}_0(T)\oplus T$$

which is the direct sum of $\mathcal{D}_0(T)$ and T. Let's define the composition law on $[T,T]\oplus T$ as follows:

$$(\forall a, b, c, d, x \in T)\quad [a,b]=-[b,a]=D_{a,b}=-D_{b,a},$$

$$[D_{a,b}, x]=-[x,D_{a,b}]=[a,b,x], \tag{1}$$

$$[D_{a,b},D_{c,d}]=D_{a,b}D_{c,d} - D_{c,d}D_{a,b}.$$

(the law is propagated on all the space by linearity).

Proposition 5.

1. For the law introduced by relations (1), the vector space: $\Gamma=[T,T]\oplus T=\mathcal{D}_0(T)\oplus T$ is a *Lie* algebra.
2. The pair $(\mathcal{D}_0(T),T)$ is symmetric, i.e. $[\mathcal{D}_0(T),T]\subseteq T$ and $[T,T]\subseteq \mathcal{D}_0(T)$.

The proof is evident.

Let $<E,(.),[-,-,-]>$ be a *Bol* algebra over $\mathbb{R}$.

Théorème. (*L. Sabinin, P. Mikheev* [15-19])

Suppose that $(E,(.),[-,-,-])$ is a *Bol* algebra over $\mathbb{R}$; then there exists a finite-dimensional *Lie* algebra Γ over the field $\mathbb{R}$, a subalgebra L in Γ, and a linear imbedding $\varphi: E \rightarrow \Gamma$ such that (after identifying $\varphi(E)$ with E):

1. $\Gamma = L \oplus E$ (direct sum of vector spaces),
2. $[[E,E],E] \subseteq E$,
3. $(\forall x, y, z \in E)$ $x.y=[x,y]_E$ and $[x,y,z]=[[x,y],z]$,

where $[x,y]$ denotes the result of commuting in Γ, and $[x,y]_E$ is the projection of vector $[x,y]$ parallelly to L onto E.

Definition 5. The vector endomorphism $\Delta: E \rightarrow E$ is said a pseudoderivation of E with the companion $d \in E$, if ([4]): $(\forall x, y, z \in E)$

1) $\Delta(x.y)=\Delta(x).y+x.\Delta(y)+[x,y,d]-(x.y).d$,
2) $\Delta[x,y,z]=[\Delta x,y,z]+[x,\Delta y,z]+[x,y,\Delta z]$.

Let D and Δ be two pseudoderivations of E, with respective companions d and δ, and let $\alpha, \beta \in \mathbb{R}$, $a, b \in E$

Proposition 6. [15-16,18]

1. $\alpha D + \beta \Delta$ is a pseudoderivation of E with companion $\alpha d + \beta \delta$.
2. $[D, \Delta] = D\Delta - \Delta D$ is a pseudoderivation of E with companion $D(\delta) - \Delta(d) - d.\delta$.
3. $D_{a,b}$ is a pseudoderivation of E with companion $a.b$.

For the proof, see for example [18].

Let E be a *Bol* algebra and define: $\mathcal{D}_0(E) = Vect_{\mathbb{R}}\{(D_{a,b}, a.b), a,b \in E\}$

and $\Gamma = \mathcal{D}_0(E) \oplus E = \{((D_{a,b}, a.b), x), a,b,x \in E\}$ (direct sum of vector spaces).

Let's introduce an internal composition law on Γ as follows:

$$\begin{cases} \left[\left((D_{a,b}, a.b), 0\right), \left((D_{c,d}, c.d), 0\right)\right] = \left(\left([D_{a,b}, D_{c,d}], D_{a,b}(c.d) - D_{c,d}(a.b) - (a.b).(c.d)\right), 0\right) \\ \left[\left((D_{a,b}, a.b), 0\right), \left((0,0), x\right)\right] = \left((0,0), D_{a,b}(x)\right) \\ \left[\left((0,0), x\right), \left((0,0), y\right)\right] = \left((D_{x,y}, x.y), 0\right) \end{cases} \quad (2)$$

(The law is propagated on Γ by linearity).

Proposition 7. [18]

1. Equipped with the composition law (2), Γ is a *Lie* algebra.
2. The pair ($Ɖ_o(E)$, E) is symmetric.
3. The vector subspace $L = Vect_R \left\{\left((D_{a,b}, ab), -ab\right), a, b \in E\right\}$ is a subalgebra of Γ.
4. $dim\, L = dim\, Ɖ_o(E)$ and $L \cap E = \{0\}$.

For the proof, see for example [18].

N.B.*

$(\forall a, b \in E)\ \left[((0,0), a), ((0,0), b)\right] = \left((D_{a,b}, a.b), 0\right) =$

$\left((D_{a,b}, a.b), -a.b\right) + \quad ((0,0), a.b)$

, i.e. $a.b = [a,b]_E$ (projection of $[a,b]$ onto E parallelly to L).

$$*(\forall a, b, c \in E),\ [[a,b],c] = \left[\left[((0,0), a), ((0,0), b)\right], ((0,0), c)\right]$$

$$= \left[\left((D_{a,b}, a.b), 0\right), ((0,0), c)\right] = \left((0,0), D_{a,b}(c)\right) = [a,b,c]$$

Consequence. Let $\Gamma = L \oplus E$ a *Lie* algebra. Suppose that (L,E) is a symmetric pair, where L is a subalgebra of Γ and E a vector subspace. Then the binary operation of the *Bol* algebra E associated with pair (L,E) is identified by some subalgebra K of Γ, such that $\Gamma = K \oplus E$ (direct sum of vector spaces). The composition laws in E are given by the following formulas:

$(\forall a, b, c \in E)$ $\qquad [a,b,c] = [[a,b], c]$

$\qquad a.b = [a,b]_E$ (projection of vector $[a,b]$ onto E parallelly to K).

3. *Lie* algebra g_2. Symmetric pair $g_2/so(4)$

The g_2 algebra is defined as being the *Lie* algebra of the exceptional compact group G_2 of automorphisms of octonions O (Cayley numbers). We'll work with a concrete matrix realization of g_2. We then fix table 1 of octonions multiplication.

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	$-e_5$	e_4	e_7	$-e_6$

e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	$-e_7$	e_6	$-e_5$	e_4
e_4	e_4	e_5	$-e_6$	e_7	$-e_0$	$-e_1$	e_2	$-e_3$
e_5	e_5	$-e_4$	$-e_7$	$-e_6$	e_1	$-e_0$	e_3	e_2
e_6	e_6	$-e_7$	e_4	e_5	$-e_2$	$-e_3$	$-e_0$	e_1
e_7	e_7	e_6	e_5	$-e_4$	e_3	$-e_2$	$-e_1$	$-e_0$

Table 1: Octonions multiplication.

The *Lie* algebra g_2 is isomorph to the algebra of derivations of the octonions $\mathbb{O}$. We'll realize g_2 using square matrixes (8×8). Let $x,y \in \mathbb{O}$; we have:

$$g_2 = \{D \in so(7) \subset so(8) : D(x.y) = Dx\,.\,y - x.Dy\}.$$

Let $\{e_i\}_{i=0}^{7}$ the canonical basis of algebra $\mathbb{O}$, adopted in table 1. Let's introduce in $so(8)$ the matrixes $\{G_{ij}: 0 \le i < j \le 7\}$, by the following relations:

$$G_{ij}e_j = e_i, \quad G_{ij}e_i = -e_j, \quad G_{ij}e_k = 0, \text{ if } k \ne i, j.$$

Proposition 8.

1. $\{G_{ij},\ 0 \le i < j \le 7\}$ is a basis of algebra $so(8)$.
2. $\{G_{ij},\ 0 < i < j \le 7\}$ is a basis of algebra $so(7)$ supposed included in $so(8)$ in a standard manner (i.e. for $D \in so(8)$, $D \in so(7)$ if and only if $De_0 = 0$).
3. $[G_{ij}, G_{k\ell}] = G_{ij}G_{k\ell} - G_{k\ell}G_{ij} = \delta_{jk}G_{i\ell} + \delta_{ik\ell}G_{jk} - \delta_{ik}G_{j\ell} - \delta_{j\ell}G_{ik}$ (3)

Proof.

1) & 2): it's obvious that $\{G_{ij},\ 0 \le i < j \le 7\}$ and $\{G_{ij},\ 0 < i < j \le 7\}$ are free generating families respectively in $so(8)$ and $so(7)$.

3) $[G_{ij}, G_{k\ell}](e_t) = G_{ij}G_{k\ell}(e_t) - G_{k\ell}G_{ij}(e_t) = G_{ij}(\delta_{\ell t}e_k - \delta_{kt}e_\ell) - G_{kl}(\delta_{tj}e_i - \delta_{ti}e_j)$

$= \delta_{lt}(\delta_{kj}e_i - \delta_{ki}e_j) - \delta_{kt}(\delta_{\ell j}e_i - \delta_{\ell i}e_j) - [\delta_{tj}(\delta_{i\ell}e_k - \delta_{ik}e_\ell) - \delta_{ti}(\delta_{\ell j}e_k - \delta_{kj}e_\ell)]$

$= (\delta_{\ell t}\delta_{kj} - \delta_{kt}\delta_{\ell j})e_i + (\delta_{kt}\delta_{\ell i} - \delta_{\ell t}\delta_{ki})e_j + (\delta_{ti}\delta_{\ell j} - \delta_{tj}\delta_{i\ell})e_k + (\delta_{tj}\delta_{ik} - \delta_{ti}\delta_{kj})e_\ell.$

On the other hand,

$(\delta_{jk}G_{i\ell} + \delta_{i\ell}G_{jk} - \delta_{ik}G_{j\ell} - \delta_{j\ell}G_{ik})(e_t) = (\delta_{\ell t}\delta_{kj} - \delta_{kt}\delta_{\ell j})e_i + (\delta_{kt}\delta_{\ell i} - \delta_{\ell t}\delta_{ki})e_j$

$+ (\delta_{ti}\delta\ell_j - \delta_{tj}\delta_{i\ell})e_k + (\delta_{tj}\delta_{ik} - \delta_{ti}\delta_{kj})e_\ell.$

Thus, $[G_{ij}, G_{k\ell}] = \delta_{jk}G_{i\ell} + \delta_{i\ell}G_{jk} - \delta_{ik}G_{j\ell} - \delta_{j\ell}G_{ik}$.

Let's introduce the linear applications: $L_i: \quad \mathbb{O} \to \mathbb{O}$ with the following formulas:

$$x \to L_i(x)$$

$$(\forall x \in \mathbb{O}) \quad L_i(x) = e_i x, \quad i = 1, \ldots, 7.$$

Lemma. In the chosen basis (table 1), matrixes L_i have the following forms:

$L_1 = -G_{01}-G_{23}+G_{45}-G_{67}$ $\quad$ $L_2 = -G_{02}+G_{13}-G_{46}-G_{57}$ $\quad$ $L_3 = -G_{03}-G_{12}+G_{47}-G_{56}$

$L_4 = -G_{04}-G_{15}+G_{26}-G_{37}$ $\quad$ $L_5 = -G_{05}+G_{14}+G_{27}+G_{36}$ $\quad$ $L_6 = -G_{06}+G_{17}-G_{24}-G_{35}$

$L_7 = -G_{07}-G_{16}-G_{25}+G_{34}$

The proof results immediately from table 1.

Proposition 9. Let D be an element of $so(8)$ such that $De_0=0$, and for each $i>0$, $De_i=\sum_{k=1}^{7} \alpha_{ik}\, e_k$. Then $D\in g_2$ if and only if $[D,L_i]= \sum_{k=1}^{7} \alpha_{ik}\, L_k$.

Proof.

$D(e_i.e_j)=De_i.e_j+e_i.De_j \Leftrightarrow D(e_i.e_j)-e_i.De_j=De_i.e_j \Leftrightarrow DL_i(e_j)-L_iD(e_j)= \sum_{k=1}^{7} \alpha_{ik}\, L_k(e_j) \Leftrightarrow$

$[D,L_i](e_j)= \sum_{k=1}^{7} \alpha_{ik}\, L_k(e_j)$,

then $[D,L_i]= \sum_{k=1}^{7} \alpha_{ik}\, L_k$.

Let A be a matrix of $so(7)\subset so(8)$; $A = \sum_{1\le i<j\le 7} a_{ij} G_{ij}$. $(Ae_0=0)$

Proposition 10. $A \in g_2 \Leftrightarrow \begin{cases} a_{17} = a_{24} + a_{35} \ (i) & a_{27} = -a_{14} - a_{36} \ (ii) \\ a_{25} = -a_{16} + a_{34} \ (iii) & a_{45} = a_{23} + a_{67} \ (iv) \\ a_{26} = a_{15} + a_{37} \ (v) & a_{46} = a_{13} - a_{57} \ (vi) \\ a_{47} = a_{12} + a_{56} \ (vii) & \end{cases}$ (4)

Proof. According to formula (3), we have:

$[G_{ij},L_1] = -[G_{ij},G_{01}]-[G_{ij},G_{23}]+[G_{ij},G_{45}]-[G_{ij},G_{67}] = -(\delta_{j0}G_{i1}+\delta_{i1}G_{j0}- \delta_{i0}G_{j1}- \delta_{j1}G_{i0})$
$-(\delta_{j2}G_{i3}+\delta_{i3}G_{j2}- \delta_{i2}G_{j3}- \delta_{j3}G_{i2})+(\delta_{j4}G_{i5}+\delta_{i5}G_{j4}- \delta_{i4}G_{j5}- \delta_{j5}G_{i4})-(\delta_{j6}G_{i7}+\delta_{i7}G_{j6}- \delta_{i6}G_{j5}- \delta_{j7}G_{i6})$,

then

$[A,L_1] =\sum_{0\le i<j\le 7} a_{ij}[G_{ij}, L_1]=a_{12}G_{02}+a_{13}G_{03}+a_{14}G_{04}+a_{15}G_{05}+a_{16}G_{06}+a_{17}G_{07}$

$+a_{13}G_{12}-a_{12}G_{13}+a_{15}G_{14}+a_{14}G_{15}+a_{17}G_{16}-a_{16}G_{17}+(a_{25}+a_{34})G_{24}+(a_{24}+a_{35})G_{25}+(a_{27}+a_{36})G_{26}$

$+(-a_{26}+a_{37})G_{27}+(-a_{24}+a_{35})G_{34}+(-a_{25}+a_{34})G_{35}+(-a_{26}+a_{37})G_{36}+(-a_{27}-a_{36})G_{37}+(a_{47}-a_{56})G_{46}$

$+(-a_{46}-a_{57})G_{47}+(a_{46}+a_{57})G_{56}+(a_{47}-a_{56})G_{57}$.

But $\{L_i\}_{i=1}^{7} \cup \{G_{ij}\}_{1\le i<j\le 7}$ is a basis in *so(8)*, then the decomposition must be unique according to this basis:

$$[A,L_1]=-a_{12}L_2-a_{13}L_3-a_{14}L_4-a_{15}L_5-a_{16}L_6-a_{17}L_7+(-a_{16}-a_{25}+a_{34})G_{24}+(-a_{17}+a_{24}+a_{35})G_{25}$$
$$+(a_{14}+a_{27}+a_{36})G_{26}+(a_{15}-a_{26}+a_{37})G_{27}+(a_{17}-a_{24}-a_{35})G_{34}+(-a_{16}-a_{25}+a_{34})G_{35}$$
$$+(a_{15}-a_{26}+a_{37})G_{36}+(-a_{14}-a_{27}-a_{36})G_{37}+(-a_{12}+a_{47}-a_{56})G_{46}+(a_{13}-a_{46}+a_{57})G_{47}$$
$$+(-a_{13}+a_{46}+a_{57})G_{56}+(-a_{12}+a_{47}-a_{56})G_{57},$$

then $[A, L_1]\in Vect\{L_i\}_{i=1}^{7}$ if and only if equalities *(i), (ii), (iii), (v), (vi)* and *(vii)* in (4) are satisfied. Operating similarly with $[A, L_2]$, we find:

$$[A,L_2]=a_{12}L_1-a_{23}L_3-a_{24}L_4-a_{25}L_5-a_{26}L_6-a_{27}L_7+(a_{16}+a_{25}-a_{34})G_{14}+(a_{17}-a_{24}-a_{35})G_{15}$$
$$+(-a_{14}-a_{27}-a_{36})G_{16}+(-a_{15}+a_{26}-a_{37})G_{17}+(a_{14}+a_{27}+a_{36})G_{34}+(a_{15}-a_{26}+a_{37})G_{35}$$
$$+(a_{16}+a_{25}-a_{34})G_{36}+(a_{17}-a_{24}-a_{35})G_{37}+(-a_{12}+a_{47}-a_{56})G_{45}+(a_{23}-a_{45}+a_{67})G_{47}$$
$$+(-a_{23}+a_{45}-a_{67})G_{56}+(a_{12}-a_{47}+a_{56})G_{67},$$

then equality (*iv*): $a_{45} = a_{23}+a_{67}$ in (4) is also satisfied.

Consequences. General form for elements of the realization matrix of g_2, in the adopted case table 1, is the following:

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{24}+a_{35}\\
0 & -a_{12} & 0 & a_{23} & a_{24} & -a_{16}+a_{34} & a_{15}+a_{37} & -a_{14}-a_{36}\\
0 & -a_{13} & -a_{23} & 0 & a_{34} & a_{35} & a_{36} & a_{37}\\
0 & -a_{14} & -a_{24} & -a_{34} & 0 & a_{23}+a_{67} & a_{13}-a_{57} & a_{12}+a_{56}\\
0 & -a_{15} & a_{16}-a_{34} & -a_{35} & -a_{23}-a_{67} & 0 & a_{56} & a_{57}\\
0 & -a_{16} & -a_{15}-a_{36} & -a_{36} & -a_{13}-a_{57} & -a_{56} & 0 & a_{67}\\
0 & -a_{24}-a_{35} & a_{14}+a_{36} & -a_{37} & -a_{12}-a_{56} & -a_{57} & -a_{67} & 0
\end{bmatrix}$$

Proposition 11.

1. *dim* $g_2=14$.
2. The following matrixes represent a basis of g_2:

$$\begin{array}{llllll}
e_1=G_{14}-G_{27} & e_2=G_{15}-G_{37} & e_3=G_{16}+G_{34} & e_4=G_{17}+G_{35} & & \\
e_5=G_{24}-G_{35} & e_6=G_{25}+G_{34} & e_7=G_{26}+G_{37} & e_8=-G_{27}+G_{36} & & \\
E_1=G_{23}-G_{67} & E_2=G_{13}+G_{57} & E_3=G_{12}+G_{47} & E_4=G_{45}+G_{67} & E_5=G_{46}-G_{57} & E_6=G_{47}+G_{56}
\end{array} \quad (5)$$

Proof.

1. Relations (4) are visibly independent and *dim* $so(7)=21$, so *dim* $g_2=14$.
2. It suffices to prove that family $\{e_i\}_{i=1}^{8} \cup \{E_i\}_{i=1}^{6}$ is free, since *dim* $g_2=14$ and the number of matrixes is *14*. Assume: $e_{j+8}=E_j$, $j=1,\ldots 6$. So $\sum_{i=1}^{14} a_i\, e_i = 0$,

   then using (5) we obtain:

$a_{11}G_{12}+a_{10}G_{13}+a_1G_{14}+a_2G_{15}+a_3G_{16}+a_4G_{17}+a_9G_{23}+a_5G_{24}+a_6G_{25}+a_7G_{26}-(a_1+a_8)G_{27}$
$+(a_3+a_6)G_{34}+(a_4+a_5)G_{35}+a_8G_{36}+(-a_2+a_7)G_{37}+a_{12}G_{45}+a_{13}G_{46}+(a_{11}+a_{14})G_{47}+a_{14}G_{56}$
$+(a_{10}-a_{13})G_{57}+(a_{12}-a_9)G_{67}=0,$

but family $\{G_{12},...,G_{67}\}$ is free, so:

$a_{11}=a_{10}=a_1=a_2=a_3=a_4=a_9=a_5=a_6=a_7=0,$

$a_1+a_8=0,\ a_3+a_6=0,\ a_4-a_5=0,\ -a_2+a_7=0,\ a_{12}=0,\ a_{13}=0,$

$a_{11}+a_{14}=0,\ a_{14}=0,\ a_{10}-a_{13}=0,\ a_{12}=0,$

hence $a_i=0,\ i=1,...,14.$

Assume $L=Vect_R\{E_1,...,E_6\}$, $E=Vect_R\{e_1,...,e_8\}$ and $d_{ij}=[e_i,e_j]=e_ie_j-e_je_i,\ 1\leq i<j\leq 8.$

Proposition 12.

1. L is a subalgebra of algebra g_2 and $dim\ E=6$.
2. The pair (L,E) is symmetric and $g_2=L\oplus E$.
3. $L=Vect_R\{d_{12}, d_{13}, d_{14}, d_{16}, d_{17}, d_{23}\}$.
4. L is isomorph to $so(4)=so(3)\oplus so(3)$.

Proof. Based on formula (3) and equalities (5), we can draw the following tables:

$[-,-]$	E_1	E_2	E_3	E_4	E_5	E_6
E_1	0	E_3-E_6	$-E_2-E_5$	0	E_6	$-E_5$
E_2	$-E_3+E_6$	0	E_1+E_4	$-E_6$	0	E_4
E_3	E_2+E_5	$-E_1-E_4$	0	$-E_5$	E_4	0
E_4	0	E_6	E_5	0	$-2E_6$	$2E_5$
E_5	$-E_6$	0	$-E_4$	$2E_6$	0	$-2E_4$
E_6	E_5	$-E_4$	0	$-2E_5$	$2E_4$	0

Table 2: Multiplication in sub-algebra L relative to base E_i.

$[-,-]$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_1	0	$-E_1-E_4$	$-E_2-E_5$	$-E_3$	$-E_3$	$-E_2$	E_1	0
e_2	E_1+E_4	0	$-E_6$	$-2E_2$	E_2	$-E_3$	0	E_1
e_3	E_2+E_5	E_6	0	$-E_4$	E_1+E_5	0	$-E_3$	$-E_2-E_5$
e_4	E_3	$2E_2$	E_4	0	0	E_1+E_4	$-E_2$	E_3-E_6
e_5	E_3	$-E_2$	$-E_1-E_5$	0	0	$-2(E_1+E_4)$	$-E_5$	E_6
e_6	E_2	E_3	0	$-E_1-E_4$	$2(E_1+E_4)$	0	$-E_6$	$-E_5$
e_7	E_1	0	E_3	E_2	E_5	E_6	0	$-2E_1$
e_8	0	$-E_1$	E_2+E_5	E_6-E_3	$-E_6$	E_5	$2E_1$	0

Table 3: Multiplication in subspace E relative to base e_i

$[-,-]$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
E_1	e_7	e_8	e_4+e_5	$-e_3+e_6$	$-e_6$	e_5	$-2e_8$	$2e_7$
E_2	$-e_6$	$-2e_4$	e_1-e_8	$2e_2$	$-e_2$	e_1	e_4	e_3-e_6
E_3	$-2(e_4+e_5)$	$-e_6$	$-e_7$	e_1	e_1	e_2	e_3	$-e_4-e_5$
E_4	$-(e_2+e_7)$	e_1-e_8	$-e_4$	e_3	$-e_6$	e_5	e_8	$-e_7$
E_5	$-e_3+e_6$	e_4	e_1-e_8	$-e_2$	$-e_7$	$-e_8$	e_5	e_6
E_6	$-(e_4+e_5)$	$-e_3$	e_2	e_1-e_8	e_8	$-e_7$	e_6	$-e_5$

Table 4: Mixed multiplication [L, E].

1. According to table 2, L is stable for composition, so L is a subalgebra of g_2.
2. According to tables 3 and 4, we have $[L,E]\subseteq E$, $[E,E]\subseteq L$, so the pair (L,E) is symmetric. It's obvious that $g_2=L\oplus E$ (direct sum of vector spaces).
3. Matrixes d_{12}, d_{13}, d_{14}, d_{16}, d_{17} and d_{23} are linearly independent; effectively, according to relations (5), we have:

$$d_{12}=[G_{14}-G_{27}, G_{15}-G_{37}] = -G_{23}-G_{45}, \quad d_{13}=[G_{14}-G_{27}, G_{16}+G_{34}] = -G_{13}-G_{46},$$
$$d_{14}=[G_{14}-G_{27}, G_{17}+G_{35}] = -G_{12}-G_{47}, \quad d_{16}=[G_{14}-G_{27}, G_{25}+G_{34}] = -G_{13}-G_{57}, \qquad (6)$$
$$d_{17}=[G_{14}-G_{27}, G_{26}+G_{37}] = G_{23}-G_{67}, \quad d_{23}=[G_{15}-G_{37}, G_{16}+G_{34}] = -G_{47}-G_{56}.$$

Suppose that:

$$\lambda_1 d_{12}+\lambda_2 d_{13}+\lambda_3 d_{14}+\lambda_4 d_{16}+\lambda_5 d_{17}+\lambda_6 d_{23}=0 \Rightarrow$$

$$(\lambda_5-\lambda_1)G_{23}-(\lambda_2+\lambda_4)G_{13}-\lambda_3 G_{12}-\lambda_2 G_{46}-(\lambda_2+\lambda_6)G_{46}-\lambda_1 G_{45}-\lambda_5 G_{67}-\lambda_6 G_{56}-\lambda_4 G_{57}=0 \Rightarrow \lambda_1=\ldots=\lambda_6=0.$$

Then:

$$Vect_R\{d_{12}, d_{13}, d_{14}, d_{16}, d_{17}, d_{23}\} = L,$$

and we also have:

$$d_{12}=-d_{35}=-d_{46}=\tfrac{1}{2}d_{56}, \quad d_{13}=d_{38}, \quad d_{14}=d_{15}=d_{26}=d_{37}, \quad d_{16}=\tfrac{1}{2}d_{24}=-d_{25}=d_{47},$$
$$d_{17}=d_{28}=-\tfrac{1}{2}d_{78}, \quad d_{23}=-d_{58}=d_{67}, \; d_{18}=d_{27}=d_{36}=d_{45}=0, \quad d_{57}=d_{68}=d_{13}-d_{16}, \qquad (7)$$
$$d_{34}=d_{12}+d_{17}, \quad d_{48}=d_{23}-d_{14}.$$

$[-,-]$	d_{12}	d_{13}	d_{14}	d_{16}	d_{17}	d_{23}
d_{12}	0	$d_{23}-d_{14}$	d_{16}	$-d_{14}$	0	$d_{16}-d_{13}$
d_{13}	$d_{14}-d_{23}$	0	d_{17}	0	$-d_{14}$	$d_{12}+d_{17}$
d_{14}	$-d_{16}$	$-d_{17}$	0	d_{12}	d_{13}	0
d_{16}	d_{14}	0	$-d_{12}$	0	$d_{23}-d_{14}$	$-d_{12}-d_{17}$
d_{17}	0	d_{14}	$-d_{13}$	$d_{14}-d_{23}$	0	$d_{16}-d_{13}$
d_{23}	$d_{13}-d_{16}$	$-d_{12}-d_{17}$	0	$d_{12}+d_{17}$	$d_{13}-d_{16}$	0

Table 5: Multiplication in sub-algebra L relative to base d_{ij}.

4. Assume that: $L_1=½(d_{12}-d_{17})$ $L_2=-½(d_{16}+d_{13})$ $L_3=\frac{d_{23}}{2}-d_{14}$

$L_4=-½(d_{12}+d_{17})$ $L_5=½(d_{16}-d_{13})$ $L_6=½d_{23}$,

We obtain the multiplication table 6, which is of *so(4)*.

	L_1	L_2	L_3	L_4	L_5	L_6
L_1	0	$-L_3$	L_2	0	0	0
L_2	L_3	0	$-L_1$	0	0	0
L_3	$-L_2$	L_1	0	0	0	0
L_4	0	0	0	0	L_6	$-L_5$
L_5	0	0	0	$-L_6$	0	L_4
L_6	0	0	0	$-L_5$	$-L_4$	0

Table 6: Multiplication in sub-algebra L relative to base L_i.

Then, L is isomorph to $so(3)\oplus so(3)$.

4. Bol Algebras of $g_2/so(4)$

Let $a=\sum_{i=1}^{8} a_i\, e_i$ an element of E. Suppose that $[d_{12},a]=\sum_{i=1}^{8} \alpha_i\, e_i$, and let D_{12} the square matrix (8×8) such that $D_{12}\overline{a}=\overline{\alpha}$, where $\overline{a}={}^{T}(a_1,\dots,a_8)$ and $\overline{\alpha}={}^{T}(\alpha_1,\dots,\alpha_8)$. Similarly, we introduce matrixes D_{13}, D_{14}, D_{16}, D_{17}, D_{23} that correspond, in the canonical basis of R^8, respectively to matrixes d_{13}, d_{14}, d_{16}, d_{17}, d_{23}; for example, $[d_{12},a]=(e_1,\dots,e_8).D_{12}\overline{a}=\sum_{i=1}^{8} \alpha_i\, e_i$.

Lemma 1. Matrixes D_{12}, D_{13}, D_{14}, D_{16}, D_{17} and D_{23} can be written as follows:

$$D_{12}=\begin{bmatrix}0&-1&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&-1&0&0&-2&0&0\\0&0&0&-1&2&0&0&0\\0&0&0&0&0&0&0&-1\\0&0&0&0&0&0&1&0\end{bmatrix}\quad D_{13}=\begin{bmatrix}0&0&-2&0&0&-1&0&0\\0&0&0&-1&1&0&0&0\\1&0&0&0&0&0&0&-1\\0&1&0&0&0&0&-1&0\\0&0&0&0&0&0&-1&0\\0&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&2&0&0&1&0&0\end{bmatrix}\quad D_{14}=\begin{bmatrix}0&0&0&-1&-1&0&0&0\\0&0&0&0&0&-1&0&0\\0&0&0&0&0&0&-1&0\\2&0&0&0&0&0&0&1\\2&0&0&0&0&0&0&1\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{bmatrix}$$

$$D_{16}=\begin{bmatrix}0&0&-1&0&0&-1&0&0\\0&0&0&-2&1&0&0&0\\0&0&0&0&0&0&0&-1\\0&2&0&0&0&0&-1&0\\0&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\end{bmatrix}\quad D_{17}=\begin{bmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&-1&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&1&0&0&1&0&0\\0&0&0&1&-1&0&0&0\\1&0&0&0&0&0&0&2\\0&1&0&0&0&0&-2&0\end{bmatrix}\quad D_{23}=\begin{bmatrix}0&0&0&-1&0&0&0&0\\0&0&-1&0&0&0&0&0\\0&1&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&1\\0&0&0&0&0&0&-1&0\\0&0&0&0&0&1&0&2\\0&0&0&1&-1&0&0&0\end{bmatrix}\qquad (8)$$

The result follows from tables 3 and 4 immediately.

For the binary composition law, according to table 5, relations (7) and proposition 7, we have:

$\cdot$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_1	0	x_1	x_2	x_3	x_3	x_4	x_5	0
e_2	$-x_1$	0	x_6	$2x_4$	$-x_4$	x_3	0	x_5
e_3	$-x_2$	$-x_6$	0	x_1+x_5	$-x_1$	0	x_3	x_2
e_4	$-x_3$	$-2x_4$	$-x_1-x_5$	0	0	$-x_1$	x_4	x_6-x_3
e_5	$-x_3$	x_4	x_1	0	0	$2x_1$	x_2-x_4	$-x_6$
e_6	$-x_4$	$-x_3$	0	x_1	$-2x_1$	0	x_6	x_2-x_4
e_7	$-x_5$	0	$-x_3$	$-x_4$	$-x_2+x_4$	$-x_6$	0	$-2x_5$
e_8	0	$-x_5$	$-x_2$	$-x_6+x_3$	x_6	$-x_2+x_4$	$2x_5$	0

Table 7: General form of the *Bol* algebra binary law.

where $x_1=e_1.e_2$, $x_2=e_1.e_3$, $x_3=e_1.e_4$, $x_4=e_1.e_6$, $x_5=e_1.e_7$, $x_6=e_2.e_3$.

Let $x_i = \sum_{j=1}^{8} x_{ij}\, e_j$ and $X_i= {}^{T}(x_{i1},\dots,x_{i8})$, $i=1,\dots 6$.

Similarly, let's designate by X_iX_j, $1\leq i<j\leq 6$, columns of coefficients of development of 15 elements $x_i.x_j$, $1\leq i<j\leq 6$, according to basis $\{e_i\}_{i=1}^{8}$.

Lemma 2. In our case, identity *6* in definition *2* is equivalent to the following matrix equations system:

1) $X_1X_2=D_{12}X_2-D_{13}X_1+X_3-X_6$ 2) $X_1X_3=D_{12}X_3-D_{14}X_1-X_4$ 3) $X_1X_4=D_{12}X_4-D_{16}X_1+X_3$

4) $X_1X_5=D_{12}X_5-D_{17}X_1$ 5) $X_1X_6=D_{12}X_6-D_{23}X_1+X_2-X_4$ 6) $X_2X_3=D_{13}X_3-D_{14}X_2-X_5$

7) $X_2X_4=D_{13}X_4-D_{16}X_2$ 8) $X_2X_5=D_{13}X_5-D_{17}X_2+X_3$ 9) $X_2X_6=D_{13}X_6-D_{23}X_2-X_1-X_5$

10) $X_3X_4=D_{14}X_4-D_{16}X_3-X_1$ 11) $X_3X_5=D_{14}X_5-D_{17}X_3-X_2$ 12) $X_3X_6=D_{14}X_6-D_{23}X_3$

13) $X_4X_5=D_{16}X_5-D_{17}X_4+X_3-X_6$ 14) $X_4X_6=D_{16}X_6-D_{23}X_4+X_1+X_5$

15) $X_5X_6=D_{17}X_6-D_{23}X_5+X_2-X_4$ (9)

In the following, we'll just treat particular case where the *Bol* algebra verifies the identity:

$$(\forall\, x, y, z, u \in E)\ \ (x.y).(z.u) = 0.$$

In this case, the system (9) is equivalent to two systems: the first one is homogeneous and linear:

1) $D_{12}X_2-D_{13}X_1+X_3-X_6=0$ 2) $D_{12}X_3-D_{14}X_1-X_4=0$ 3) $D_{12}X_4-D_{16}X_1+X_3=0$

4) $D_{12}X_5-D_{17}X_1=0$ 5) $D_{12}X_6-D_{23}X_1+X_2-X_4=0$ 6) $D_{13}X_3-D_{14}X_2-X_5=0$

7) $D_{13}X_4-D_{16}X_2=0$ 8) $D_{13}X_5-D_{17}X_2+X_3=0$ 9) $D_{13}X_6-D_{23}X_2-X_1-X_5=0$

10) $D_{14}X_4-D_{16}X_3-X_1=0$ 11) $D_{14}X_5-D_{17}X_3-X_2=0$ 12) $D_{14}X_6-D_{23}X_3=0$

13) $D_{16}X_5-D_{17}X_4+X_3-X_6=0$ 14) $D_{16}X_6-D_{23}X_4+X_1+X_5=0$

15) $D_{17}X_6-D_{23}X_5+X_2-X_4=0,$ (10)

and the second one is homogeneous and "cubic":

$$X_iX_j=0, \quad 1\leq i<j\leq 6. \qquad (10bis)$$

The real solution that verifies the two systems simultaneously is trivial. But the extension of the scalars field gives us several remarkable complex solutions.

Proposition 13. The general solution for system (10) is given by:

$X_1={}^T(-\alpha_3, \alpha_4, 0, 0, \alpha_2\text{-}2\alpha_7, -\alpha_1\text{-}2\alpha_8, \alpha_4\text{-}\alpha_5, -\alpha_6)$

$X_2={}^T(2\alpha_2\text{-}\alpha_7, -\alpha_8, 2\alpha_4\text{-}\alpha_5, \alpha_3+\alpha_6, \alpha_6, 0, -\alpha_1\text{-}\alpha_8, -2\alpha_2+\alpha_7)$

$X_3={}^T(2\alpha_1+\alpha_8, -\alpha_7, \alpha_6, \alpha_4+\alpha_5, \alpha_4+\alpha_5, \alpha_3, -\alpha_2, 0)$

$X_4={}^T(\alpha_2\text{-}\alpha_7, \alpha_1\text{-}\alpha_8, \alpha_4\text{-}\alpha_5, 2\alpha_3+\alpha_6, 0, \alpha_5, 0, -\alpha_2)$

$X_5={}^T(0, 0, \alpha_1, -\alpha_2, -\alpha_2+\alpha_7, \alpha_8, -\alpha_4+2\alpha_5, \alpha_3+2\alpha_6)$

$X_6={}^T(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8), \quad \alpha_i\in R, i=1,\dots,8. \qquad (11\text{-}0)$

Proof. Matrixes D_{23}, $D_{12}\text{-}D_{17}$, $D_{13}+D_{16}$, $D_{13}\text{-}D_{16}$ are invertible; equation (10-12) gives:

$$X_3 = D_{23}^{-1}.D_{14}.X_6 \qquad (11\text{-}1)$$

Equations 10-5 and 10-15 imply $(D_{12}\text{-}D_{17})X_6\text{-}D_{23}(X_1\text{-}X_5)=0$, hence:

$$X_1 - X_5 = D_{23}^{-1}.(D_{12} - D_{17}).X_6 \qquad (11\text{-}2)$$

Then, $X_1 = X_5 + D_{23}^{-1}.(D_{12} - D_{17}).X_6$;

replacing X_1 by its value from equation 10-4, we find:

$$D_{12}X_5 - D_{17}(X_5 + D_{23}^{-1}(D_{12} - D_{17})X_6) = (D_{12} - D_{17})X_5 - D_{17}D_{23}^{-1}(D_{12} - D_{17})X_6 = 0,$$

hence:

$$X_5 = (D_{12} - D_{17})^{-1}D_{17}D_{23}^{-1}(D_{12} - D_{17})X_6 \qquad (11\text{-}3)$$

and

$$X_1 = [(D_{12} - D_{17})^{-1}D_{17}D_{23}^{-1}(D_{12} - D_{17}) + D_{23}^{-1}(D_{12} - D_{17})]X_6 \qquad (11\text{-}4)$$

Equations 10-9 and 10-14 imply: $(D_{13}+D_{16})X_6\text{-}D_{23}(X_2+X_4)=0$;

hence:

$$X_2 + X_4 = D_{23}^{-1}(D_{13} + D_{16})X_6 \qquad (11\text{-}5)$$

But equation 10-9 implies

$$X_2=D_{23}^{-1}.(D_{13}X_6\text{-}X_1\text{-}X_5),$$

i.e. $$X_2 = D_{23}^{-1}[D_{13} - 2(D_{12} - D_{17})^{-1}D_{17}D_{23}^{-1}(D_{12} - D_{17}) - D_{23}^{-1}(D_{12} - D_{17})]X_6 \qquad (11\text{-}6)$$

And then:

$$X_4 = [-D_{23}^{-1}(D_{13} - 2(D_{12} - D_{17})^{-1}D_{17}D_{23}^{-1}(D_{12} - D_{17}) + D_{23}^{-1}(D_{12} - D_{17}))$$
$$+D_{23}^{-1}(D_{13} + D_{16})]X_6 \quad (11\text{-}7)$$

We can easily verify that columns (11-0) given by equations 11-1, 11-3, 11-4, 11-6, 11-7 represent the general solution of the linear system (10).

According to proposition 7, it exists a subalgebra K of $g_2=L\oplus E$ such that $g_2=K\oplus E$, $(K\cap E=\{0\})$ and $(\forall\, a,b \in E)\;\; a.b = [a,b]_E$ (projection of $[a,b]$ onto E parallelly to K). Then K is generated by a basis of the following from:

$$\{P_1=d_{12}+x_1,\; P_2=d_{13}+x_2,\; P_3=d_{14}+x_3,\; P_4=d_{16}+x_4,\; P_5=d_{17}+x_5,\; P_6=d_{23}+x_6\},$$

where:

$dim\ K=6,\ x_1=e_1.e_2,\ x_2=e_1.e_3,\ x_3=e_1.e_4,\ x_4=e_1.e_6,\ x_5=e_1.e_7,\ x_6=e_2.e_3.$

By requiring stability of vector space $Vect_R\{P_j\}_{j=1,\dots 6}$ for commutation, i.e.:

$$[P_k,P_\ell] = \sum_{j=1}^{6} \lambda_{k\ell j} P_j,\ 1\le k<\ell\le 6,$$

we find that all the solutions verifying simultaneously systems (10) and (10bis) are complex except the trivial solution. Among the found solutions, we provide the following list of monoparametric families:

<u>Algebras 1/1bis</u>

$$\begin{cases} x_1 = e_1.e_2 = \mp i\alpha e_1 - \alpha e_6 \quad (\alpha \neq 0) \\ x_2 = e_1.e_3 = -\alpha e_2 \pm i\alpha e_4 \\ x_3 = e_1.e_4 = -\alpha e_1 \pm i\alpha e_6 \\ x_4 = e_1.e_6 = -2\alpha e_2 \pm 2i\alpha e_4 \\ x_5 = e_1.e_7 = -\alpha e_3 + \alpha e_6 \pm i\alpha e_8 \\ x_6 = e_2.e_3 = -\alpha e_1 \pm i\alpha e_3 + \alpha e_8 \end{cases} \quad (12\text{-}1)$$

<u>Algebras 2/2bis</u>

$$\begin{cases} x_1 = \alpha e_2 \mp i\alpha e_6 \quad (\alpha \neq 0) \\ x_2 = \alpha e_3 \mp i\alpha e_7 \\ x_3 = \pm 2i\alpha e_1 + 2\alpha e_4 + 2\alpha e_5 \\ x_4 = \pm i\alpha e_2 + \alpha e_6 \\ x_5 = \pm i\alpha e_3 + \alpha e_7 \\ x_6 = \pm i\alpha e_1 + \alpha e_4 + \alpha e_5 \end{cases} \quad (12\text{-}2)$$

<u>Algebras 3/3bis</u>

$$\begin{cases} x_1 = \pm i\alpha e_1 - \alpha e_5 \mp i\alpha e_8 \quad (\alpha \neq 0) \\ x_2 = \alpha e_1 \pm i\alpha e_5 - \alpha e_8 \\ x_3 = -\alpha e_2 \pm i\alpha e_3 \mp i\alpha e_6 - \alpha e_7 \\ x_4 = \mp i\alpha e_4 - \alpha e_8 \\ x_5 = -\alpha e_4 \pm i\alpha e_8 \\ x_6 = \alpha e_2 \mp i\alpha e_3 \pm i\alpha e_6 + \alpha e_7 \end{cases} \quad (12\text{-}3)$$

<u>Algebras 4/4bis</u>

$$\begin{cases} x_1 = \mp 2i\alpha e_5 - 2\alpha e_6 \quad (\alpha \neq 0) \\ x_2 = \mp i\alpha e_1 - \alpha e_2 - \alpha e_7 \pm i\alpha e_8 \\ x_3 = \alpha e_1 \mp i\alpha e_2 \\ x_4 = \mp i\alpha e_1 - \alpha e_2 \\ x_5 = \pm i\alpha e_5 + \alpha e_6 \\ x_6 = \pm i\alpha e_7 + \alpha e_8 \end{cases} \quad (12\text{-}4)$$

<u>Algebras 5/5bis</u>

<u>Algebras 6/6bis</u>

$$\begin{cases} x_1 = \alpha e_2 \pm i\alpha e_5 + \alpha e_7 \quad (\alpha \neq 0) \\ x_2 = \pm 2i\alpha e_1 + 2\alpha e_3 \mp 2i\alpha e_8 \\ x_3 = \alpha e_4 + \alpha e_5 \mp i\alpha e_7 \\ x_4 = \pm i\alpha e_1 + \alpha e_3 \mp i\alpha e_8 \\ x_5 = \mp i\alpha e_4 \mp i\alpha e_5 - \alpha e_7 \\ x_6 = \pm i\alpha e_2 + \alpha e_4 \end{cases} \tag{12-5}$$

$$\begin{cases} x_1 = \mp i\alpha e_7 - \alpha e_8 \quad (\alpha \neq 0) \\ x_2 = \mp i\alpha e_3 + \alpha e_4 + \alpha e_5 \\ x_3 = \alpha e_3 \pm i\alpha e_4 \pm i\alpha e_5 \\ x_4 = \mp i\alpha e_3 + \alpha e_4 \pm i\alpha e_6 \\ x_5 = \pm 2i\alpha e_7 + 2\alpha e_8 \\ x_6 = \pm i\alpha e_5 + \alpha e_6 \end{cases} \tag{12-6}$$

<u>Algebras 7/7bis</u>

$$\begin{cases} x_1 = -\frac{\alpha}{\sqrt{2}} e_1 \pm \alpha i \frac{\sqrt{3}}{2} e_2 \mp i\alpha\sqrt{6} e_5 \ -\frac{5\alpha}{2}\ e_6 \quad (\alpha \neq 0) \\ x_2 = \mp i\alpha\sqrt{\frac{3}{2}} e_1 - \alpha e_2 \pm \alpha i \frac{\sqrt{3}}{2} e_3 + \frac{\alpha}{\sqrt{2}} e_4 - \frac{3\alpha}{2} e_7 \pm i\alpha\sqrt{\frac{3}{2}} e_8 \\ x_3 = 2\alpha e_1 \mp \alpha i\sqrt{\frac{3}{2}} e_2 \pm i\alpha\sqrt{3} e_4 \pm i\alpha\sqrt{3} e_5 + \frac{\alpha}{\sqrt{2}} e_6 \\ x_4 = \mp i\alpha\sqrt{\frac{3}{2}} e_1 - \frac{\alpha}{2} e_2 + \alpha\sqrt{2} e_4 \pm \alpha i \frac{\sqrt{3}}{2} e_6 \\ x_5 = \frac{\alpha}{2} e_3 \pm i\alpha\sqrt{\frac{3}{2}} e_5 + \alpha e_6 \pm \alpha i \frac{\sqrt{3}}{2} e_7 + \frac{\alpha}{\sqrt{2}} e_8 \\ x_6 = \frac{\alpha}{2} e_1 + \frac{\alpha}{\sqrt{2}} e_3 \pm \frac{\alpha}{2} i\sqrt{3} e_4 \pm \frac{\alpha}{2} i\sqrt{3} e_5 \pm i\alpha\sqrt{\frac{3}{2}} e_7 + \alpha e_8 \end{cases} \tag{12-7}$$

<u>Algebras 8/8bis</u>

$$\begin{cases} x_1 = \frac{\alpha}{\sqrt{2}} e_1 \pm \alpha i \frac{\sqrt{3}}{2} e_2 \pm i\alpha\sqrt{6} e_5 \ -\frac{5\alpha}{2}\ e_6 \quad (\alpha \neq 0) \\ x_2 = \pm i\alpha\sqrt{\frac{3}{2}} e_1 - \alpha e_2 \pm \alpha i \frac{\sqrt{3}}{2} e_3 - \frac{\alpha}{\sqrt{2}} e_4 - \frac{3\alpha}{2} e_7 \mp i\alpha\sqrt{\frac{3}{2}} e_8 \\ x_3 = 2\alpha e_1 \pm \alpha i\sqrt{\frac{3}{2}} e_2 \pm i\alpha\sqrt{3} e_4 \pm i\alpha\sqrt{3} e_5 - \frac{\alpha}{\sqrt{2}} e_6 \\ x_4 = \pm i\alpha\sqrt{\frac{3}{2}} e_1 - \frac{\alpha}{2} e_2 - \alpha\sqrt{2} e_4 \pm \alpha i \frac{\sqrt{3}}{2} e_6 \\ x_5 = \frac{\alpha}{2} e_3 \mp i\alpha\sqrt{\frac{3}{2}} e_5 + \alpha e_6 \pm \alpha i \frac{\sqrt{3}}{2} e_7 - \frac{\alpha}{\sqrt{2}} e_8 \\ x_6 = \frac{\alpha}{2} e_1 - \frac{\alpha}{\sqrt{2}} e_3 \pm \frac{\alpha}{2} i\sqrt{3} e_4 \pm \frac{\alpha}{2} i\sqrt{3} e_5 \mp i\alpha\sqrt{\frac{3}{2}} e_7 + \alpha e_8 \end{cases} \tag{12-8}$$

<u>Algebras 9/9bis</u>

$$\begin{cases} x_1 = -\alpha e_1 \mp \frac{\alpha}{\sqrt{2}} ie_2 \pm \frac{3\alpha}{\sqrt{2}} ie_5 - 2\alpha e_6 \mp \frac{\alpha}{\sqrt{2}} i\, e_7 + \alpha e_8 \quad (\alpha \neq 0) \\ x_2 = -\alpha e_2 \mp \alpha i\sqrt{2} e_3 - \alpha e_5 - \alpha e_7 \\ x_3 = \alpha e_1 \pm \alpha i\sqrt{2} e_2 - \alpha e_3 \mp \frac{\alpha}{\sqrt{2}} ie_4 \mp \frac{\alpha}{\sqrt{2}} ie_5 + \alpha e_6 \pm \frac{\alpha}{\sqrt{2}} ie_7 \\ x_4 = \pm \frac{\alpha}{\sqrt{2}} ie_1 - \alpha e_2 \mp \frac{\alpha}{\sqrt{2}} ie_3 + \alpha e_4 \pm \frac{\alpha}{\sqrt{2}} ie_8 \\ x_5 = \pm \frac{\alpha}{\sqrt{2}} ie_4 \mp \frac{\alpha}{\sqrt{2}} ie_5 + \alpha e_6 \pm \frac{\alpha}{\sqrt{2}} ie_7 - \alpha e_8 \\ x_6 = \mp \frac{\alpha}{\sqrt{2}} ie_2 + \alpha e_3 \mp \frac{\alpha}{\sqrt{2}} ie_4 - \alpha e_6 \mp i\alpha\sqrt{2} e_7 + \alpha e_8 \end{cases} \tag{12-9}$$

Algebras 10/10bis

$$\begin{cases} x_1 = \alpha e_1 \pm \frac{\alpha}{\sqrt{2}} ie_2 \pm \frac{3\alpha}{\sqrt{2}} ie_5 - 2\alpha e_6 \pm \frac{\alpha}{\sqrt{2}} i\, e_7 - \alpha e_8 \quad (\alpha \neq 0) \\ x_2 = -\alpha e_2 \pm \alpha i\sqrt{2} e_3 + \alpha e_5 - \alpha e_7 \\ x_3 = \alpha e_1 \pm \alpha i\sqrt{2} e_2 + \alpha e_3 \pm \frac{\alpha}{\sqrt{2}} ie_4 \pm \frac{\alpha}{\sqrt{2}} ie_5 - \alpha e_6 \pm \frac{\alpha}{\sqrt{2}} ie_7 \\ x_4 = \pm \frac{\alpha}{\sqrt{2}} ie_1 - \alpha e_2 \pm \frac{\alpha}{\sqrt{2}} ie_3 - \alpha e_4 \pm \frac{\alpha}{\sqrt{2}} ie_8 \\ x_5 = \pm \frac{\alpha}{\sqrt{2}} ie_4 \mp \frac{\alpha}{\sqrt{2}} ie_5 + \alpha e_6 \mp \frac{\alpha}{\sqrt{2}} ie_7 + \alpha e_8 \\ x_6 = \mp \frac{\alpha}{\sqrt{2}} ie_2 - \alpha e_3 \pm \frac{\alpha}{\sqrt{2}} ie_4 + \alpha e_6 \mp i\alpha\sqrt{2} e_7 + \alpha e_8 \end{cases} \tag{12-10}$$

Summary and conclusion

It appears to us that these algebras represent mathematical objects which require special studying. In this paper, we've addressed the issue of *Bol* algebras generated by the symmetric space $g_2/so(4)$ and verifying an additional identity of the form $(x.y).(z.u) = 0$, that we've introduced because it seems to us that this identity can model in some way the concept of "disparity" encountered in the theory of elementary particles. It is well known that g_2 is the *Lie* algebra of G_2 automorphisms group of octonions system frequently used in various fields of research. We've provide a list of 20 mono-parametric families of complex *Bol* algebras. But this list is not exhaustive of all possible algebras in the considered case.

Bol algebras generalize those of *Lie* and of *Mal'cev*; so based on information relative to the multiplicity of application domains for theories of *Lie* and *Mal'cev* (see for example [3,23]), we can guess with large credibility that the obtained results are probably applicable in mechanics, quantic, elementary particle theories, relativist physics and in space-time theory.

Our future publications will be dedicated to the completion of the list of obtained algebras, as well as to the study of each element in this list.

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