AMSE JOURNALS –2015-Series: Modelling D; Vol. 36; N° 1; pp 32-46
Submitted June 2014; Revised Dec. 26, 2014; Accepted March 15, 2015

Optimum Taxation Policy Using Maximum-Entropy Method

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Abstract: In this paper we use Entropy in Economics to minimize the income inequality in our society. For minimization we take Entropy type constraints. Mainly we use Entropy method introduced by Shannon. Income inequality and taxation policy are two parts of Economics, which are strongly correlative. In this paper it is shown that income inequality can be reduced by suitable choice of taxation policy. This taxation policy can be applied in both discrete and continuous cases. We proof this theory in two section –first in discrete case and second in continuous case.

Keywords: Taxation Policy, Optimization, Income Inequality, Entropy, Lagrangian Constraint.

1. Introduction

Income inequality among countries in the world rise sharply. It is a big challenge to the researchers to find a suitable process to minimize this inequality. Entropy optimization technique is a beautiful method to solve this problem. So we first discuss the relation between income inequality and entropy and then minimize this income inequality by entropy.

2. Income inequality and entropy

The concept of inequality arises in various contexts and there is considerable interest in its measurement. Besides economics, in political science and sociology also inequalities are measure using various indices like voting strength tax structure etc. The measurement of species diversity in ecology is essentially a problem of measuring equality. We concentrate our thinking in income inequality in the field of economics and try to solve this income inequality by entropy.

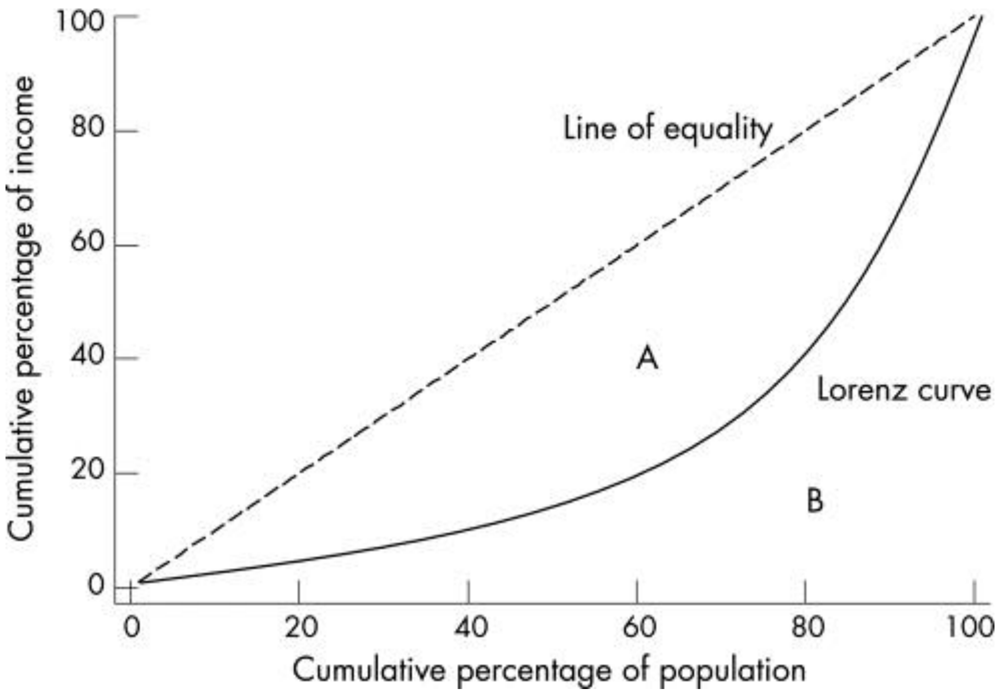

Fig 1. The **Lorenz curve**

The Lorenz curve is a graphical representation of the cumulative distribution function of the empirical probability distribution of wealth or income in economics and was developed by Max .O. Lorenz for representing inequality of the wealth distribution.

The curve is a graph showing the proportion of overall income or wealth assumed by the bottom *x*% of the people, although this is not rigorously true for a finite population. It is often used to represent income distribution, where it shows for the bottom *x*% of households, what percentage (*y*%) of the total income they have. The percentage of households is plotted on the *x*-axis, the percentage of income on the *y*-axis. It can also be used to show distribution of assets. In such use, many economists consider it to be a measure of social inequality.

Measurement of income inequality is discussed and surveyed by many authors [1,2,3]. A rational approach to a general measure of inequality (not necessarily of income wealth) is first due to Dalton [2]. According to Dalton [2] a function ϕ is said to be a measure of inequality (better an index of inequality) if it satisfies the conditions [9]:

(i) For any two vectors x = $(x_{1,}, x_2, \ldots\ldots x_n)$
And y = $(y_1, y_2, \ldots\ldots, y_n)$

$x \prec y \Rightarrow \phi(x) \le \phi(y)$

i.e. ϕ should be Schur-convex

(ii) $x \prec y$ and x is not a permutation of y

$$\Rightarrow \qquad \phi(x) < \phi(y).$$

i.e. ϕ - should be strictly Schur-convex.

The notation $x \prec y$ implies that the arguments of x are 'more equal' than those of y. these conditions were first formulated by Dalton [3] although they are hinted at or are implicit in the works of Lorenze and Pigou. Again if ϕ be a measure of inequality, then the function ψ defined for all x such that

$\sum_{i=1}^{n} x_i \neq 0$ by

$$\psi(x) = \phi\left(\frac{x_1}{\sum_{i=1}^{n} x_i}, \frac{x_2}{\sum_{i=1}^{n} x_i}, \ldots\ldots \frac{x_n}{\sum_{i=1}^{n} x_i}\right)$$

is also a measure of inequality satisfying Dalton's conditions. For measure of equality or species diversity in biology, that a maximum be achieved when all the arguments are equal, so in (i) and (ii) Schur-concavity should replace Schur-convexity.

A common measure of equality of unnormed distribution x = $(x_{1,}, x_2, \ldots\ldots x_n) \geq 0$ (negative taken measures of inequality of x) considered in econometrics by Lorenz, Pigou, Dalton and other are the functions of the form:

$$\phi \quad (x_1, x_2, \ldots\ldots\ldots\ldots\ldots, x_n) = H\left(\frac{x_1}{\sum_{i=1}^{n} x_i}, \frac{x_2}{\sum_{i=1}^{n} x_i}, \ldots\ldots \frac{x_n}{\sum_{i=1}^{n} x_i} \right)$$

Where H is some suitable entropy function. Different measure of inequality can be obtained for different forms of entropy functions H. this is a brief account of the interrelations between the concept of entropy and the measure of inequality as developed by Dalton [2] and others and this is valid for all type of the system.

3. Taxation policy (discrete case)

Let us consider the income distribution of individuals of a population or society. Suppose a society consisting of n income earners with income x_i (i =1,2,…. n). It is assumed that x_i are non-negative and that at least some of them are positive, so that both of the total income $\sum_{i=1}^{n} x_i = X$ and per capita personal income $\overline{X} = (\sum_{i=1}^{n} x_i / n)$ are positive. The income share of i-th individual is his/her share of the total personal income:

$$p_i = \frac{x_i}{\sum_{i=1}^{n} x_i} = \frac{x_i}{n\overline{X}} \qquad (1)$$

His/her population share is his/her share of the total population, which is simply 1/n for each individual. Now we define the measure of income inequality as the expected information of the measure, which transforms the population shares into the income shares:

$$H = \sum_{i=1}^{n} p_i \ln(p_i /(1/n)) = \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{\overline{\overline{X}}} \ln(\frac{x_i}{\overline{\overline{X}}}) \qquad (2)$$

Now replacing $\overline{X}$ by $\frac{X}{n}$, the expression (2) can be reduced to the form, H= $\frac{1}{n}\sum_{i=1}^{n}\frac{nx_i}{X}\ln(nx_i/X)$

$$=\sum_{i=1}^{n}(\frac{x_i}{X}\ln(n)+\frac{x_i}{X}\ln(\frac{x_i}{X}))$$

$$= \ln(n) + \sum_{i=1}^{n}\frac{x_i}{X}\ln(\frac{x_i}{X})$$

$$= \ln(n) - \sum_{i=1}^{n}-\frac{x_i}{X}\ln(\frac{x_i}{X}) \qquad (3)$$

The second term of the right hand side of (3) is the form of Shannon-Entropy

$$S= -\sum_{i=1}^{n}\frac{x_i}{X}\ln(\frac{x_i}{X}) = -\sum_{i=1}^{n}p_i \ln p_i \qquad (4)$$

The individual share $p_i=\frac{x_i}{X}$ also satisfying the conditions $p_i \geq 0$, ($\forall$ i=1,2,………,n) and $\sum_{i=1}^{n}p_i=1$ defines a probability distribution and the Shannon entropy S measures the diversity of the probability distribution $\{p_1, p_2, \ldots, p_n\}$

Again

$$L=-\sum_{i=1}^{n}p_i \ln p_i - k(\sum_{i=1}^{n}p_i - 1) \quad \text{where k is a constant.} \qquad (5)$$

Now differentiating with respect to $p_1, p_2, \ldots, p_n$ we get

$-(1+\ln p_i) - k = 0$, i=1,2,………,n

or, $\ln p_i = -1 - k$

or, $p_i = e^{-1-k} = Me^{-k}$ (where $M=e^{-1}$=constant)

$\because \sum_{i=1}^{n}p_i = 1$

$$M\sum_{i=1}^{n} e^{-k} = 1$$

Or, $Mne^{-k}=1$

Or, $M=e^k/n$

$\therefore p_i = \frac{1}{n}$, $\forall$ i i.e. maximum value is reached when $p_1 = p_2 = \ldots\ldots\ldots\ldots = p_n = \frac{1}{n}$ i.e. when all the income earners have the same income

$x_1 = x_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = x_n = \frac{1}{n}$

we can then write H as

$$H = S_{\max} - S = \ln(n) - \sum_{i=1}^{n}[-\frac{x_i}{X}\ln(\frac{x_i}{X})] \qquad (6)$$

From (6) we see that to reduce the income inequality we have to increase the value of the entropy S. For the optimal reduction of the income inequality we have to maximize the entropy S subject to some constraints or policies determined by the Government.

3. Optimal taxation with maximum entropy approach (discrete case)

As stated before let $x_1, x_2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, x_n$ be the income of the n individuals in a population and $X = \sum_{i=1}^{n} x_i$ be the total income of the population. Let f (x_i) be the certain taxation policy such that the income charged from a person is x_i f(x_i) whose income is x_i .

Let us assumed that nobody is charged more tax than his/her income and there is no negative taxation or subsides.

So we have $0 \leq f(x_i) \leq 1$, i=1,2,…………….n (7)

We must have f(x_i) to be an increasing function of x_i if we want to reduce the income inequality through taxation. Let a person whose income is x_i have the real income [x_i - x_i f(x_i)] after taxation .we also assume the fair taxation policy : $x_1 - x_1 f(x_1) \leq x_2 - x_2 f(x_2) \leq \ldots\ldots\ldots\ldots \leq [x_n - x_n f(x_n)]$

When $x_i \leq x_{i+1}$,i=1, 2,……………….,n-1 (8)

So that after taxation the richer does not become poorer. Then to minimize the income inequality is to maximize the taxation entropy:

$$S = -\sum_{i=1}^{n} \frac{x_i - x_i f(x_i)}{\sum_{i=1}^{n}[x_i - x_i f(x_i)]} \ln \frac{x_i - x_i f(x_i)}{\sum_{i=1}^{n}[x_i - x_i f(x_i)]} \qquad (9)$$

Or equivalently

$$S = -\sum_{i=1}^{n} [x_i - x_i f(x_i)] \ln [x_i - x_i f(x_i)] \qquad (10)$$

Subject to constraints

$$0 \leq f(x_i) \leq 1 \qquad (i = 1,2,3,\ldots\ldots\ldots\ldots,n) \qquad (11)$$

And $$\sum_{i=1}^{n} x_i f(x_i) = T, \; T < X \qquad (12)$$

Where T is the fixed income tax revenue.

So the problem is

To maximize $$S = -\sum_{i=1}^{n} x_i \ln (x_i q_i)$$

i.e. to maximize $$S = -\sum_{i=1}^{n} y_i \ln y_i \qquad (13)$$

where $q_i = 1 - f(x_i)$, $y_i = x_i q_i$, $i = 1,2,\ldots\ldots\ldots\ldots,n$

subject to constraints :

$$0 \leq 1 - q_i \leq 1 \qquad (i = 1,2,3,\ldots\ldots\ldots\ldots,n)$$

i.e. $$0 \leq 1 - \frac{y_i}{x_i} \leq 1$$

i.e. $$0 \leq x_i - y_i \leq x_i$$

i.e. $$-x_i \leq -y_i \leq 0$$

i.e. $$x_i \geq y_i \geq 0$$

i.e. $$0 \leq y_i \leq x_i \qquad (14)$$

And

$$\sum_{i=1}^{n} x_i [1 - q_i] = T, \quad T < X$$

or, $$\sum_{i=1}^{n} x_i \left[1 - \frac{y_i}{x_i}\right] = T$$

or, $$\sum_{i=1}^{n} [x_i - y_i] = T$$

or, $$\sum_{i=1}^{n} y_i = X - T \qquad (15)$$

To solve the inequality constraints we shall follow this technique:

Let us first ignore the inequality constraints (14) and consider the Lagrangian

$$L(\bar{y},k) = -\sum_{i=1}^{n} y_i \ln y_i \quad -k \left[\sum_{i=1}^{n} y_i - (X - T) \right] \qquad (16)$$

Now , $\frac{\partial L}{\partial y_i} = 0$ gives

$-[\ln y_i + 1] - k = 0$, $i = 1,2,\ldots\ldots,n$

or, $\ln y_i = -1 - k$

or, $y_i = e^{-k-1} = \lambda$ (say)

now the equation (15) gives

$$\sum_{i=1}^{n} \lambda = X - T$$

or, $\lambda = \frac{X - T}{n}$

or, $y_i = \frac{X - T}{n}$, ($i = 1,2,3,\ldots\ldots\ldots\ldots\ldots,n$) (17)

but this may not satisfy the first constraint (14) unless we allow subsides.

i) Now, if $\frac{X - T}{n} \le x_i$ ($i = 1,2,3,\ldots\ldots\ldots\ldots\ldots,n$)

Then $[x_i - x_i f(x_i)] = \frac{X - T}{n}$

so that after paying taxes, every body has the same income.

ii) We see that , $\frac{X - T}{n} > x_j$ for some j then we will make the inequality constraints $y_j \le x_j$ into an equality i.e. $y_j = x_j$ so that the Lagrangian in this step will be

$$L(\bar{y},k) = -\sum_{i=1}^{n} y_i \ln y_i \quad -k \left[\sum_{i \ne j} y_i + c_j - (X - T) \right] \qquad (18)$$

Then $\frac{\partial L}{\partial y_i} = 0$ $i \ne j$ gives

$-[\ln y_i + 1] - k = 0$

or, $\ln y_i = -1 - k$

or, $y_i = e^{-k-1} = \lambda_i$ (say) , $i \ne j$ (19)

Now $\sum_{i \ne j} y_i + x_j = X - T$

or, $\lambda_i (n-1) + x_j = X - T$

or, $\lambda_i = \dfrac{X - T - x_j}{n-1}$ (20)

So, in this case

$$y_i = \frac{X - T - x_j}{n-1} \quad , i \neq j \qquad (21)$$
$$= \quad x_j \quad i=j$$

now if in this step $y_i = \dfrac{X - T - x_j}{n-1} \leq x_i$, $i \neq j$

then the taxation will be

$$x_i f(x_i) = x_i - \frac{X - T - x_j}{n-1}, \quad , i \neq j$$
$$= \quad 0 \quad , \quad , i=j$$

Implying that every person except one will be left with income $\dfrac{X - T - x_j}{n-1}$ each while for the j-th person it is x_j .

iii) Again if in this step we see that, when $\dfrac{X - T - x_j}{n-1} > x_r$ for some r then, we will make two inequalities $y_j \leq x_j$ and $y_r \leq x_r$ into equalities

$y_j = x_j$ and $y_r = x_r$ (22)

So, in this step our problem is equivalent to the maximization of $-\sum_{i=1}^{n} y_i \ln y_i$ subject to the constraints:

$y_j = x_j$ and $y_r = x_r$ (23)

$\sum_{i=1}^{n} y_i = X - T$ (24)

So that the Lagrangian is this step will be

$$L(\bar{y}, k) = -\sum_{i=1}^{n} y_i \ln y_i - k \left[\sum_{i \neq j, i \neq r} y_i + x_j + x_r - (X - T) \right]$$

Then $\dfrac{\partial L}{\partial y_i} = 0$ $i \neq j$,r gives

$-[\ln y_i + 1] - k = 0$

or, $\ln y_i = -1 - k$

or, $y_i = e^{-k-1} = \lambda_i$ (say) , $i \neq j$,r or, $\sum_{i \neq j,r} y_i + x_j + x_r = X - T$

or, $\lambda_i (n-2) + x_j + x_r = X - T$

or, $\lambda_i = \frac{X - T - x_j - x_r}{n-2}$

This leads to

$$
\begin{aligned}
y_i &= \frac{X - T - x_j - x_r}{n-2} \quad , \quad i \neq j, r \\
&= x_j \quad , \quad i = j \qquad (25) \\
&= x_r \quad , i = r
\end{aligned}
$$

if (14) holds , then after paying taxes every person except two will be left with income $\frac{X - T - x_j - x_r}{n-2}$ each while for j-th and r-th person it is x_j and x_r respectively.

Then the taxation will be

$$
\begin{aligned}
x_i f(x_i) &= x_i - \frac{X - T - x_j - x_r}{n-2}, \quad i \neq j, r \qquad (26) \\
&= 0 \quad , \quad i = j, r
\end{aligned}
$$

but if (14) does not hold for some i then we are to make three inequalities into equalities and we have to proceed in this way as long as necessary.

5. Taxation policy (continuous case)

Let z be the income of any individual and let m be the average income of all individuals, then the corresponding measure of inequality is given by

$$H = \int_0^\infty \left(\frac{z}{m} \ln \frac{z}{m}\right) \varphi(z) dz \qquad (27)$$

So that income inequality is measured by the expected value of the product of relative income and the logarithm of the relative income where relative income is the ratio of the income to its expected value. Here φ (z) is the density function. Equation (27) satisfy the

conditions $z \geq 0$ and $\int_0^\infty \frac{z}{m}\varphi(z)dz = 1$. (28)

Then the Lagrangian L become

$$L=\int_0^\infty \left(\frac{z}{m}\ln\frac{z}{m}\right)\varphi(z)dz - \lambda\left(\int_0^\infty \frac{z}{m}\varphi(z)dz - 1\right) \quad (29)$$

Where λ is a constant.

Using Lagrange method and Calculus of Variations we get

$[-1-\ln\frac{z}{m} - \lambda]\varphi(z) = 0$

or ,$-1-\ln\frac{z}{m} - \lambda = 0$

or , $\ln\frac{z}{m} = -(1+\lambda)$

or , $\frac{z}{m} = e^{-1-\lambda} = A\, e^{-\lambda}$ [where A is a constant] (30)

Again we have

$\int_0^\infty \frac{z}{m}\varphi(z)dz = 1$

or, $\int_0^\infty Ae^{-\lambda}\varphi(z)dz = 1$

or, $A\, e^{-\lambda}\int_0^\infty \varphi(z)dz = 1$

or, $A\, e^{-\lambda}.1 = 1$ $\left[\because \int_0^\infty \varphi(z)dz = 1\right]$

or, $A = e^{\lambda}$

$\therefore \frac{z}{m} = A\, e^{-\lambda} = e^{\lambda} e^{-\lambda} = 1$

$\therefore z = m$ (31)

i.e. maximum entropy is reached when all the income earners have the same income and income is equal to the average income of all individuals.

$\therefore$ Entropic approach gives the same answer as in the discrete case.

6. Optimal taxation with maximum entropy approach (continuous case)

As stated before, let z be the income of any individual and f(z) be an increasing function of z which determine the taxation policies. Also let φ(z) be the density function for the income distribution. We have now to choose f(z) so as to maximize

$$\int_0^\infty \{\frac{z - zf(z)}{E[z - zf(z)]} \ln \frac{z - zf(z)}{E[z - zf(z)]}\} \, \varphi(z) \, dz \tag{32}$$

Where zf(z) be the taxation of each individuals and E[z-zf(z)] be the average income of all individuals after taxation.

Since we want to minimize income inequality, we want to maximize entropy subject to some constraints. The constraint we shall consider is that the government revenue from income tax should remain fixed. There can be an infinite number of taxation policies, each of which will yield the same revenue to the government. Out of this our object is to choose that policy which will maximize the entropy or minimize the income inequality.

To keep the expected value of taxes fixed we have

$$E(z) = \int_0^\infty z\varphi(z)dz = X \quad \text{and } E[zf(z)] = \int_0^\infty z f(z)\varphi(z)dz = T \tag{33}$$

Then we have to maximize

$$\int_0^\infty \{\frac{z - zf(z)}{X - T} \ln \frac{z - zf(z)}{X - T}\} \, \varphi(z) \, dz$$

$$= \frac{1}{X-T}\{\int_0^\infty [z - zf(z)]\ln[z - zf(z)]\varphi(z)\,dz - \int_0^\infty [z - zf(z)]\ln[X - T]\varphi(z)\,dz\}$$

$$= \frac{1}{X-T}\{\int_0^\infty [z - zf(z)]\ln[z - zf(z)]\varphi(z)\,dz - \ln(X-T) \qquad (34)$$

subject to (33) .

Using Lagrange's method and calculus of variations, this gives

$$[-1-\ln[z-zf(z)]-\lambda]\varphi(z) = 0 \qquad (35)$$

Where λ is a Lagrange multiplier.

Or, $[\ln[z-zf(z)] = -1-\lambda$

Or, $z-zf(z) = e^{-1-\lambda} = k$ (say) (36)

From (33) and (36)

$$\int_0^\infty (z-k)\varphi(z)dz = T$$

Or, X-k = T

Or, k = X - T (37)

But this will be negative when z < X - T .
As such, we consider the taxation function

$zf(z) = 0$ when $z < X - T$ (38)

$= z - k'$ when $z \geq X - T$

So that from (33)

$$\int_0^\infty (z-k')\varphi(z)dz = T \qquad (39)$$

Knowing $\varphi(z)$, this determines k'. Since

$$\int_0^\infty (z-k)\varphi(z)dz = X - k = T \qquad (40)$$

We have $z-k' > z-k$

Or $k' < k = X-T$ (41)

So that the density function in (39) ≥ 0.

8. Discussion

Thus in the continuous case also, the heuristic algorithm gives the same answer as in the discrete case viz the following: “Do not charge any income tax from persons with income above this, charge in such a way that the net income after income tax deduction is a constant”.

The Lorenz Curve also change it shape after taxation. It become near to the line of equality after taxation. So any income tax from persons minimize the income inequality.

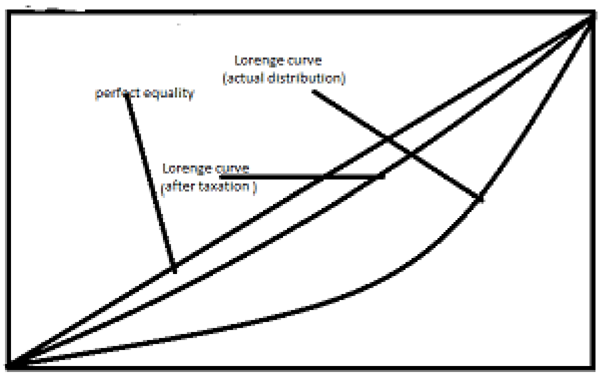

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Figure 2. Lorenz curve after taxation

7. Conclusion

It is a new idea using maximum-entropy approach to taxation policy which is futuraristic in nature and a real life application with some minor restrictions. By maximum-entropy we can find the proper tax of a person such that the income inequality of a country will minimize. One of the main ways in which governments can influence income distribution is through the system of cash benefits and personal taxes. This paper focuses on the effects of taxes and benefits on income distribution**.** Taxes tend to be progressive, in the sense that people with higher incomes pay a higher proportion of their income in tax. Benefits may be targeted at the poor or, even if fl at rate, they will narrow the proportional difference between the incomes of the rich and the poor. When benefits are paid to people in particular circumstances, these tend to be correlated with low income or greater needs (such as childhood, disability, etc.) or are benefits that are specifically intended to replace income from work (unemployment benefit, pension).

Acknowledgment

The authors are highly thankful to the referees on providing constructive suggestion which have improved the earlier version of this paper.

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