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MinxEnt Principle and its Application in Regional and Urban Planning

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Abstract

Kullback- Leibler's measure (*D (p:q)*) of cross entropy has been used for the modelling in regional and urban planning. Considering the regional and urban population distributions to be arithmetic and geometric, we have shown that the nature of distribution is independent of Kullback- Leibler's measure. Using these results we find D (p:q) for Binomial, Poisson and Negative binomial distributions which comes out to be a convex function of the new parameter and a minimum value of zero when the new parameter coincides with the original parameter**.** We have shown the difference of results obtained from MinxEnt and MaxEnt principle.

Keywords : Regional and urban planning, Entropy, Kullback-Leibler's cross entropy, MaxEnt, MinxEnt.

1. Introduction

The perception of entropy was first presented by Clausius in 1864 [1] into thermodynamics in the middle of the nineteenth century, an advanced used in a dissimilar procedure by L. Boltzmann in his revolutionary effort on the kinetic theory of gases in 1872 [2]. We are interested in information-theoretic entropy rather than thermodynamic entropy. Both the concepts are related but for the application of problems such as regional and urban planning, urban transport system, particularly, the traffic network and flow to meet future demand [3-5], traffic signal optimization [6], different statistical problem [7-10], econometric application [11] etc., the former is suitable for our exposition.

In the last two decades there has been a great interest in generalizing Shanon entropy principle [12] exploring its consequences to physics and other problems. The subsequent discovery of Jayens [13] arose with the connection in statistical mechanics deriving probability distributions by maximizing the Shanon measure which is known to be Jayne's maximum entropy principle. Shannon's information-theoretic entropy was widespread by Kullback and Leibler [14] in 1951 by way of relative entropy or cross entropy or the divergence measure between two probability distributions. Kullback's MaxEnt principle can be demonstrated from Jayne's maximum entropy principle but the Kullback's minimum cross entropy principle is based on entirely two different concepts:

- Kullback-Leibler's (K-L) measure
- Minimization of the K-L measure subject to specified linear moment constraints.

The Kullback-Leibler's divergence measure deals with the detachment between two probability density distributions. This divergence is similarly identified as information divergence and relative entropy. The reputation of detachment between probability distributions ascends as of the character they show in the complications of inference and discrimination. The perception of detachment between two probability distributions was primarily established by Mahalanobis [15] in 1936. Since then numerous forms of detachment measures have been established in the works [11]. A perception thoroughly connected to the one of detachment method is that of divergence measures grounded on the inkling of information-theoretic entropy major announced in communication theory by Shannon [12] and advanced by Wiener [16].

Kullback-Leibner method can be used in multidisciplinary research. Kullback-Leibner information can be applied as a basis in ecological studies [17]. Cross entropy method has been used for signal optimization as shown in Ref. [6], spectral density function [18],iterative image reconstruction [19], econometric applications [11], estimating social accounting matrix to find an efficient and cost-effective way to incorporate and reconcile information from a variety of sources, including data from prior years [20] etc. We present the K-L method for urban and regional planning. These models are of great importance in two ways. Firstly, urban and regional models using entropy optimization method is the key to achieve a scientific understanding of cities and regions. Secondly, an increasingly importance in several associated planning of existing urban and regional socioeconomic problems. So many works have been done in urban and regional planning using Shanon entropy [12], Jayne's entropy [13] etc.

The purpose of the present work is to study MinxEnt principle from a theoretical point of view for the application in rural and regional planning using Kullback – Leibler's measure of cross entropy method.

The paper is organized as follows: In Section 2 we present the Modelling on Urban and Regional planning and Kullback – Leibler's measure of cross entropy method and subsequently we discuss the properties of K-L measure. In Section 3 we discuss the results. Finally we conclude in Section 5.

2. Modelling on Urban and Regional planning

Let $p_1,p_2,\ldots,p_n$ be the amounts of inhabitants of a city, existing in n inhabited colonies and let the conforming comprehensive cost of breathing be $c_1,c_2,\ldots,c_n$ individually. Each comprehensive cost embraces the cost of transportable from the colony to the Central Business District (C.B.D.) as shown in Fig. 1. Now enchanting into explanation the capabilities and civic facilities, we may like the populations to be circulated in the ratios $q_1,q_2,\dots,q_n$. These will give magnitudes that can live in these colonies affording to some fixed norms. In this case, we should like to minimize,

$$\sum_{i=1}^{n} p_i \ln(\frac{p_i}{q_i}) \text{ , Subject to } \sum_{i=1}^{n} p_i = 1 \text{ and } \sum_{i=1}^{n} p_i c_i = c \text{ .}$$

Using Lagrange's method we obtain the solution

$$f(\mu) \equiv \frac{\sum_{i=1}^{n} q_i c_i e^{-\mu c_i}}{\sum_{i=1}^{n} q_i e^{-\mu c_i}} - c = 0 \qquad [1]$$

where, μ is Lagrange's undetermined multiplier. The detailed description of the solution is given in Appendix A.

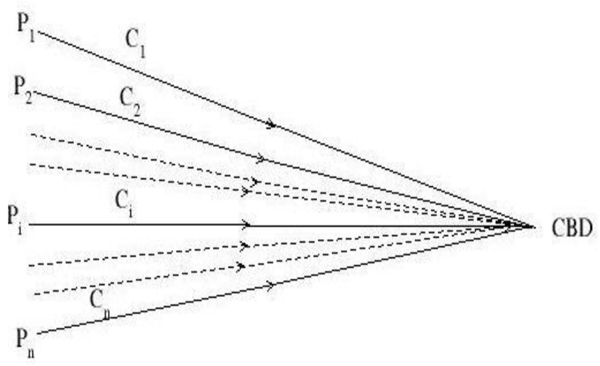

Fig. 1: Proportions of $p_1,p_2,\dots,p_n$ Populations living in n colonies and travelling to CBD.

We consider Kullback-Leibler's cross entropy method to perform the entropy optimization technique in the modelling of rural and urban planning. In the next section we present a brief description of Kullback-Leibler's cross entropy method along with its properties.

2.1 Kullback – Leibler's measure of cross entropy method:

Kullback's Minimum cross entropy principle (MinxEnt), is an entropy optimization technique. The MinxEnt principle depends on two facts, first of all, the cross entropy of a probability distribution P with respect to the other probability distribution Q and that of an priori probability distribution.

The Kullback-Leibler measure of cross entropy is the most fundamental measure among all such possible measures.

Let $p = (p_1, p_2, ..., p_n)$ and $q = (q_1, q_2, ..., q_n)$ be two probability distributions. Then the Kullback-Leibler measure is defined as

$$D(p:q) = \sum_{i=0}^{n} p_i \ln(\frac{p_i}{q_i})$$

where we assume that whenever $q_i = 0$, the corresponding p_i is also zero. We define $0\ln(\frac{0}{0}) \equiv 0$.

2.2 Properties of Kullback-Leibler (K-L) Measure:

i) $D(p:q) \geq 0$.

ii) $D(p:q) = 0$ iff p = q.

iii) $D(p:q)$ should be a convex function of $p_1, p_2, ..., p_n$.

iv) When $D(p:q)$ is minimized subject to known linear constraints using Lagrange's method, none of the resulting minimizing probabilities should be negative.

v) $D(p:q)$ is a continuous function of $p_1, p_2, ..., p_n$ and $(q_1, q_2, ..., q_n)$.

vi) $D(p:q)$ is permutationally symmetric.

3 Result and Discussion

For rural and regional planning, we present different distribution functions to suite the situation. In this paper we present two different distributions depending on q using the aforementioned model and from this result we explain the interesting nature of p.

Case I: q is the Arithmetic Distribution.

CaseII: q is Geometric Distribution.

3.1 When q is the Arithmetic Distribution

Here we shall find out the probability distribution of *p* when *q* is the arithmetic distribution i.e. $\{k'a, k'(a+d), k'(a+2d), ..., k'(a+nd)\}$ using MinxEnt principle provided mean *m* is given.

We have to minimize $\sum_{i=0}^{n} p_i \ln(\frac{p_i}{q_i})$ subject to $\sum_{i=0}^{n} p_i = 1$ and $\sum_{i=0}^{n} ip_i = m$.

Now the Lagrangian L is given by

$$L \equiv \sum_{i=1}^{n} p_i \ln(\frac{p_i}{q_i}) + \lambda(\sum_{i=0}^{n} p_i - 1) + \mu(\sum_{i=0}^{n} ip_i - m)$$

$$\frac{\delta L}{\delta p_i} = \ln(\frac{p_i}{q_i}) + p_i \cdot \frac{q_i}{p_i} \cdot \frac{1}{q_i} + \lambda + \mu i$$

Now, $\frac{\delta L}{\delta p_i} = 0$ gives,

$$\Rightarrow \ln(\frac{p_i}{q_i}) + 1 + \lambda + \mu i = 0$$

$$\Rightarrow \ln(\frac{p_i}{q_i}) = -1 - \lambda - \mu i$$

$$\Rightarrow p_i = q_i.e^{-(1+\lambda)}.e^{-\mu i}$$

$$\Rightarrow p_i = k.q_i.b^i$$

where, $k = e^{-(1+\lambda)}$

and, $b = e^{-\mu}$

k and b can be determined from,

$$kk'\sum_{i=0}^{n}(a+id).b^i = 1 \text{ and } kk'\sum_{i=0}^{n} i(a+id).b^i = m$$

So, the probability distribution of p is given as

$\{k.k'a, k.k'(a+d).b, k.k'(a+2d).b^2, ..., k.k'(a+nd).b^n\}$ i.e. also a Arithmetic distribution.

3.2 When q is the Geometric Distribution

Here we shall find out the probability distribution of p when q is the arithmetic distribution i.e. $\left\{k', k'c, k'c^2, ..., k'c^n\right\}$ using MinxEnt principle provided mean m is given.

\We have to minimize $\sum_{i=0}^{n} p_i \ln(\frac{p_i}{q_i})$ subject to $\sum_{i=0}^{n} p_i = 1$ and $\sum_{i=0}^{n} ip_i = m$.

Now the Lagrangian L is given by

$$L \equiv \sum_{i=1}^{n} p_i \ln(\frac{p_i}{q_i}) + \lambda(\sum_{i=0}^{n} p_i - 1) + \mu(\sum_{i=0}^{n} ip_i - m)$$

Now, $\frac{\delta L}{\delta p_i} = 0$ gives,

$$\Rightarrow \ln(\frac{p_i}{q_i}) + 1 + \lambda + \mu i = 0$$

$$\Rightarrow \ln(\frac{p_i}{q_i}) = -1 - \lambda - \mu i$$

$$\Rightarrow p_i = q_i.e^{-(1+\lambda)}.e^{-\mu i}$$

$$\Rightarrow p_i = k.q_i.b^i$$

Where, $k = e^{-(1+\lambda)}$
and, $b = e^{-\mu}$

k and *b* can be determined from,

$$kk'\sum_{i=0}^{n} c^i b^i = 1 \text{ and } kk'\sum_{i=0}^{n} i c^i b^i = m$$

So, the probability distribution of p is given as

$\{k.k', k.k'c.b, k.k'c^2.b^2, ..., k.k'c^n.b^n\}$ i.e. also a Geometric distribution.

Corollary1: *Values of D (p:q) for different priori distribution and D (p:q) as convex function of the new parameter and a minimum value of zero when the new parameter coincides with the original parameter***.**

It can be inferred from the above two results that when distribution q is arithmetic, p comes out to be arithmetic. Same is true for geometric distribution. Now in the next part we calculate the values of D(p:q) for the two different distribution (q): (i) Binomial Distribution and (ii) Poisson Distribution and (iii) negative Binomial Distribution taking into consideration the fact that p will be Binomial and Poisson distribution, negative Binomial Distribution respectively.

3.3 For Binomial Distribution

Here the priori probability distribution *q* is given by ${}^nC_r p_0^r (1-p_0)^{n-r}$ where r=0, 1,…, n. So, by MinxEnt principle the posteriori probability distribution *p* is given by ${}^nC_r p^r (1-p)^{n-r}$ where r=0, 1,…, n.

So,

$$D(p:q) = \sum_{i=0}^{n} p_i \ln(\frac{p_i}{q_i})$$
$$= \sum_{i=0}^{n} {}^nC_i p^i (1-p)^{n-i} \ln\{\frac{{}^nC_i p^i (1-p)^{n-i}}{{}^nC_i p_0^i (1-p_0)^{n-i}}\}$$
$$= \sum_{i=0}^{n} {}^nC_i p^i (1-p)^{n-i} \ln(\frac{p}{p_0})^i + \sum_{i=0}^{n} {}^nC_i p^i (1-p)^{n-i} \ln(\frac{1-p}{1-p_0})^{n-i}$$
$$= \ln(\frac{p}{p_0})\sum_{i=0}^{n} i\, {}^nC_i p^i (1-p)^{n-i} + \ln(\frac{1-p}{1-p_0})\sum_{i=0}^{n} (n-i)\, {}^nC_i p^i (1-p)^{n-i}$$
$$= np\ln(\frac{p}{p_0}) + (n-np)\ln(\frac{1-p}{1-p_0}) \qquad [2]$$

Now, we shall show that D (p:q) as a convex function of the new parameter i.e. p:

From Equation (2), we get,

$$D(p:q) = np\ln(\frac{p}{p_0}) + (n-np)\ln(\frac{1-p}{1-p_0})$$

$$\frac{\delta D}{\delta p} = n\ln(\frac{p}{p_0}) + np.\frac{p_0}{p}.\frac{1}{p_0} + (-n)\ln(\frac{1-p}{1-p_0}) + (n-np)(\frac{1-p_0}{1-p})(\frac{-1}{1-p_0})$$

$$= n\ln(\frac{p}{p_0}) - n\ln(\frac{1-p}{1-p_0}) + n - n$$

$$= n\ln(\frac{p}{p_0}) - n\ln(\frac{1-p}{1-p_0}) \qquad [3]$$

Similarly,

$$\frac{\partial^2 D}{\partial p^2} = n.\frac{p_0}{p}.\frac{1}{p_0} - n.(\frac{1-p_0}{1-p})(\frac{-1}{1-p_0})$$

$$= \frac{n}{p} - \frac{n}{1-p}$$

$$= \frac{n}{p(1-p)}$$

This is positive.

So, that *D (p:q)*as a convex function of the new parameter i.e. *p*.

Minimum value of D (p:q):

Minimum value of *D (p:q)* obtained by making $\frac{\delta D}{\delta p} = 0$.

$$\Rightarrow n\ln(\frac{p}{p_0}) - n\ln(\frac{1-p}{1-p_0}) = 0$$

$$\Rightarrow n\ln(\frac{p}{p_0}) = n\ln(\frac{1-p}{1-p_0})$$

$$\Rightarrow \ln(\frac{p}{p_0}) = \ln(\frac{1-p}{1-p_0})$$

$$\Rightarrow \frac{p}{p_0} = \frac{1-p}{1-p_0}$$

$$\Rightarrow p - pp_0 = p_0 - pp_0$$

$$\Rightarrow p = p_0$$

i.e. minimum value will be obtained only when the new parameter coincides with the original parameter and the minimum value is

From Equation (2), we get,

D (p:q)

$$= np\ln(\frac{p}{p_0}) + (n-np)\ln(\frac{1-p}{1-p_0})$$
$$= np.\ln 1 + (n-np).\ln 1$$
$$= 0$$

3.4 For Poisson Distribution

Here the priori probability distribution q is given by $e^{-m_0}.\frac{m_0^i}{i!}, i = 0,1,2,...$ So, the posterior probability distribution p is given by $e^{-m}.\frac{m^i}{i!}, i = 0,1,2,...$

So,

$$D(p:q) = \sum_{i=0}^{\infty} p_i \ln(\frac{p_i}{q_i})$$
$$= \sum_{i=0}^{\infty} e^{-m}.\frac{m^i}{i!}\ln(\frac{e^{-m}.\frac{m^i}{i!}}{e^{-m_0}.\frac{m_0^i}{i!}})$$
$$= \sum_{i=0}^{\infty} e^{-m}.\frac{m^i}{i!}\ln(e^{m_0-m}) + \sum_{i=0}^{\infty} e^{-m}.\frac{m^i}{i!}\ln(\frac{m}{m_0})^i$$
$$= (m_0 - m) + m\ln(\frac{m}{m_0}) \qquad [4]$$

Now, we shall show that D (p:q) as a convex function of the new parameter i.e. m:

From Equation (4), we get,

$$D(p:q) = (m_0 - m) + m\ln(\frac{m}{m_0})$$
$$\frac{\delta D}{\delta m} = -1 + \ln(\frac{m}{m_0}) + m.\frac{m_0}{m}.\frac{1}{m_0}$$
$$\frac{\delta D}{\delta m} = -1 + \ln(\frac{m}{m_0}) + 1$$
$$\frac{\delta D}{\delta m} = \ln(\frac{m}{m_0})$$

Similarly,

$$\frac{\partial^2 D}{\partial m^2} = \frac{m_0}{m} \cdot \frac{1}{m_0}$$

$$\frac{\partial^2 D}{\partial m^2} = \frac{1}{m}$$

This is a positive.

So, that *D (p:q)*as a convex function of the new parameter i.e. *m*.

Minimum value of D (p:q):

Minimum value of *D (p:q)* obtained by making $\frac{\delta D}{\delta m} = 0$

$$\Rightarrow \ln(\frac{m}{m_0}) = 0$$

$$\Rightarrow \frac{m}{m_0} = e^0$$

$$\Rightarrow \frac{m}{m_0} = 1$$

$$\Rightarrow m = m_0$$

i.e. minimum value will be obtained only when the new parameter coincides with the original parameter and the minimum value is

From Equation(4), we get,

$$D(p:q) = (m_0 - m_0) + m\ln(\frac{m_0}{m_0})$$

$$= 0 + m\ln(1)$$

$$= 0$$

3.5 For Negative Binomial Distribution

Here the priori probability distribution *q* is given by $\frac{(n+i-1)!}{(n-1)!i!}(\frac{n}{m_0})^n(1-\frac{n}{m_0})^i$,i=0,1,2,…

So, the posteriori probability distribution *p* is given by $\frac{(n+i-1)!}{(n-1)!i!}(\frac{n}{m})^n(1-\frac{n}{m})^i$,i=0,1,2,....

So,

$$D(p:q)=\sum_{i=0}^{\infty}\frac{(n+i-1)!}{(n-1)!i!}(\frac{n}{m})^n(1-\frac{n}{m})^i\ln(\frac{\frac{(n+i-1)!}{(n-1)!i!}(\frac{n}{m})^n(1-\frac{n}{m})^i}{\frac{(n+i-1)!}{(n-1)!i!}(\frac{n}{m_0})^n(1-\frac{n}{m_0})^i})$$

$$=\sum_{i=0}^{\infty}\frac{(n+i-1)!}{(n-1)!i!}(\frac{n}{m})^n(1-\frac{n}{m})^i\ln(\frac{m_0}{m})^n+\sum_{i=0}^{\infty}\frac{(n+i-1)!}{(n-1)!i!}(\frac{n}{m})^n(1-\frac{n}{m})^i i\ln(\frac{(1-\frac{n}{m})^i}{(1-\frac{n}{m_0})^i})$$

$$=n\ln(\frac{m_0}{m})+\frac{m}{n}\ln(\frac{1-\frac{n}{m}}{1-\frac{n}{m_0}}) \qquad [5]$$

4 Conclusion

In conclusion, we can assume, without loss of generality that $c_1,c_2,\ldots,c_n$ are arranged in ascending order of magnitude so that

$$c_1\le c_2\le c_3\le \ldots\le c_n \qquad [6]$$

Using Equations (5) and (6),

$$f(-\infty)=c_n-c,\ f(0)=\sum_{i=1}^{n}q_ic_i-c=c'-c,\ f(\infty)=c_1-c$$

$$f'(\mu)=-\frac{\sum_{i=1}^{n}q_ie^{-\mu c_i}\sum_{i=1}^{n}q_ic_i.c_ie^{-\mu c_i}-\sum_{i=1}^{n}q_ic_ie^{-\mu c_i}.\sum_{i=1}^{n}q_ic_ie^{-\mu c_i}}{(\sum_{i=1}^{n}q_ie^{-\mu c_i})^2}$$

$$\Rightarrow f'(\mu)=-\frac{\sum_{i=1}^{n}q_ie^{-\mu c_i}\sum_{i=1}^{n}q_ic_i^2e^{-\mu c_i}-(\sum_{i=1}^{n}q_ic_ie^{-\mu c_i})^2}{(\sum_{i=1}^{n}q_ie^{-\mu c_i})^2}\le 0 \qquad [7]$$

$f'(\mu)$=0 iff $c_1=c_2=c_3=\ldots=c_n$ which we will assume not the case, so $f'(\mu)<0$, i.e. $f'(\mu)$ is a decreasing function of μ.So, the maximum value of $f(\mu)$ is c_n-c and minimum value is c_1-c.

Now, we will discuss about the characteristics of the roots of the equation $f(\mu)=0$.

1. $f(\mu)=0$ has no real root if $c<c_1$ *or* $c>c_n$.
2. $f(\mu)=0$ has a positive root if $c_1<c<c'$.
3. $f(\mu)=0$ has a negative root if $c'<c<c_n$.
4. $f(\mu)=0$ is satisfied by μ=0 if c=c'.

Now, we will find the minimum cross entropy using MinxEnt principle. This is expressed in Fig. 2.

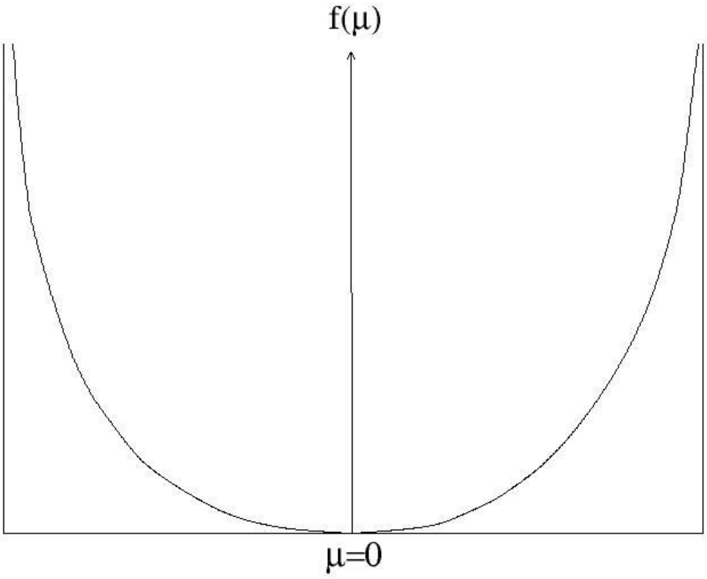

Fig. 2: Characteristics of function $f(\mu)$

$$D(p:q)=\sum_{i=1}^{n} p_i \ln(\frac{p_i}{q_i})$$

$$\Rightarrow D(p:q)=\sum_{i=1}^{n}\frac{q_i e^{-\mu c_i}}{\sum_{i=1}^{n} q_i e^{-\mu c_i}}.\ln(\frac{q_i e^{-\mu c_i}}{q_i.\sum_{i=1}^{n} q_i e^{-\mu c_i}})$$

$$\Rightarrow D(p:q)=\sum_{i=1}^{n}\frac{q_i e^{-\mu c_i}}{\sum_{i=1}^{n} q_i e^{-\mu c_i}}(-\mu c_i)-\sum_{i=1}^{n}\frac{q_i e^{-\mu c_i}}{\sum_{i=1}^{n} q_i e^{-\mu c_i}}\ln(\sum_{i=1}^{n} q_i e^{-\mu c_i})$$

$$\Rightarrow D(p:q)=-\mu\sum_{i=1}^{n} p_i c_i-\sum_{i=1}^{n} p_i \ln(\sum_{i=1}^{n} q_i e^{-\mu c_i})$$

$$\Rightarrow D(p:q)=-\mu c-\ln(\sum_{i=1}^{n} q_i e^{-\mu c_i}) \qquad [8]$$

Now, taking differentiation, we get,

$$\frac{\delta D}{\delta c}=-\mu-c.\frac{\delta\mu}{\delta c}-\frac{\sum_{i=1}^{n}-c_i q_i e^{-\mu c_i}}{\sum_{i=1}^{n} q_i e^{-\mu c_i}}.\frac{\delta\mu}{\delta c}$$

$$\Rightarrow \frac{\delta D}{\delta c}=-\mu-c\frac{\delta\mu}{\delta c}+c\frac{\delta\mu}{\delta c}$$

$$\Rightarrow \frac{\delta D}{\delta c}=-\mu \qquad [9]$$

$$\Rightarrow \frac{\partial^2 D}{\partial c^2}=-\frac{\delta\mu}{\delta c} \qquad [10]$$

Now, from Equation (10) we can conclude that *D(p:q)* is a convex function of *c*, as from Equation (9) we have $\frac{\delta\mu}{\delta c}<0$.

D(p:q) will be minimum only when q_i is uniform distribution i.e. $\{\frac{1}{n},\frac{1}{n},\frac{1}{n},...,\frac{1}{n}\}$ that's happen only when $c=\overline{c}$ i.e. when $\{\frac{1}{n},\frac{1}{n},\frac{1}{n},...,\frac{1}{n}\}$.

Now the minimum value of *D(p:q)*is given by,

$$D(p:q)_{\min} = -\mu\frac{e^{-\frac{\mu}{n}}.\frac{n}{n^2}}{e^{-\frac{\mu}{n}}} - \ln(e^{-\frac{\mu}{n}})$$

$$\Rightarrow D(p:q)_{\min} = -\frac{\mu}{n}+\frac{\mu}{n}$$

$$\Rightarrow D(p:q)_{\min} = 0$$

From Equation (10) and from the characteristics of the roots of the equation $f(\mu)=0$ we have the following results:

$c_1 < c < c' \Rightarrow \mu > 0 \Rightarrow \frac{\delta D}{\delta c} < 0 \Rightarrow D$ is an decreasing function of *c*.

$c' < c < c_n \Rightarrow \mu < 0 \Rightarrow \frac{\delta D}{\delta c} > 0 \Rightarrow D$ is an increasing function of *c*.

In Table 1 we present the comparison of results obtained by MaxEnt and MinxEnt principles and it is observed that maximum value is obtained only when uniform distribution is occurred whereas minimum value is obtained for uniform distribution whereas minimum value is obtained for uniform distribution of q_i and c_i.

Table 1: Comparison of Results obtained by using MaxEnt and MinxEnt Principles

Using MaxEnt Principle	**Using MinxEnt principle**
p_i is independent of q_i.	p_i is dependent of q_i.
Maximum Entropy S is a concave function of c.	Minimum Entropy is a convex function of c.
Maximum Entropy is ln(n).	Minimum value is 0.
Maximum value obtained only when Uniform distribution is occurred.	Minimum value obtained on Uniform distribution of q_i and c_i.

Appendix A: Calculation of the Lagrangian

Lagrangian L is given by,

$$L \equiv \sum_{i=1}^{n} p_i \ln(\frac{p_i}{q_i}) + \lambda(\sum_{i=1}^{n} p_i - 1) + \mu(\sum_{i=1}^{n} p_i c_i - c)$$

$$\Rightarrow \frac{\delta L}{\delta p_i} = \ln(\frac{p_i}{q_i}) + p_i . \frac{q_i}{p_i} . \frac{1}{q_i} + \lambda + \mu c_i$$

$$\Rightarrow \frac{\delta L}{\delta p_i} = 1 + \lambda + \mu c_i + \ln(\frac{p_i}{q_i})$$

Now, making $\frac{\delta L}{\delta p_i}$ =0, we get,

$$\Rightarrow 1 + \lambda + \mu c_i + \ln(\frac{p_i}{q_i}) = 0$$

$$\Rightarrow \ln(\frac{p_i}{q_i}) = -1 - \lambda - \mu c_i$$

$$\Rightarrow p_i = q_i . e^{(-1-\lambda-\mu c_i)}$$

$$\Rightarrow p_i = q_i . e^{-(1+\lambda)} . e^{-\mu c_i}$$

Now using constraints, we get,

$$\sum_{i=1}^{n} p_i = 1$$

$$\Rightarrow \sum_{i=1}^{n} q_i . e^{-(1+\lambda)} . e^{-\mu c_i} = 1$$

$$\Rightarrow \sum_{i=1}^{n} q_i e^{-\mu c_i} = e^{(1+\lambda)}$$

Again,

$$\sum_{i=1}^{n} p_i c_i = c$$

$$\Rightarrow \sum_{i=1}^{n} q_i . e^{-(1+\lambda)} . e^{-\mu c_i} . c_i = c$$

$$\Rightarrow \frac{\sum_{i=1}^{n} q_i c_i e^{-\mu c_i}}{\sum_{i=1}^{n} q_i e^{-\mu c_i}} = c$$

μ can be obtained from the equation,

$$f(\mu) \equiv \frac{\sum_{i=1}^{n} q_i c_i e^{-\mu c_i}}{\sum_{i=1}^{n} q_i e^{-\mu c_i}} - c = 0 \qquad [A1]$$

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