Mathematical Modeling and Simulation of a Diffusion Process in the Human Bloodstream

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Abstract

Models describing the variation of concentration of solute in the bloodstream over distance, x and time, t as blood solution moves at constant velocity through the blood vessel were formulated from first principle. The models are solved both analytically and numerically.

The models were used to simulate the diffusion process in human bloodstream, determine the parameters that affect the flux density and how the system responds to unit change in these parameters. It was discovered that the concentration gradient between the bloodstream and its surrounding fluid decreases exponentially as the blood solution flows through the vessel at a constant velocity, v. Hence concentration of solute in the bloodstream approaches that of the solute in its surrounding fluid as the distance, x travelled becomes very large. The concentration of the solute in the bloodstream also decreases with an increase in time, t and approaches that in the surrounding fluid as time spent gets large. The flux density (the time rate at which solute diffuses per unit area) decreases with distance, x since the concentration gradient decreases with distance, x and approach zero (no diffusion) as the concentration gradient approach zero. The flux density becomes zero (an equilibrium state) when the concentration of solute in bloodstream is equal to the concentration in the surrounding fluid.

It was also noticed that the flux density decreases with an increase in the velocity of the solution in the blood vessel. Hence the flux density depends majorly on the solute concentration gradient between the bloodstream and its surrounding fluid, the distance, x travelled by the blood solution in the vessel, time, t spent and the velocity of the blood solution.

The usefulness of this work have been identified to include but not limited to nutrient uptake from the blood, infections by pathogenic secretions, dialysis, drug action, gaseous exchange etc.

Key words: Models, diffusion, concentration gradient, flux density and solute

1.0 Introduction.

The objective of this project is to successfully model a diffusion process in the human blood stream that will account for such phenomena as drug absorption, exchange of gases, nutrient uptake, etc. The model will then be simulated to provide a better understanding of the influence of the factors affecting diffusion.

The dependence of life processes on diffusion mechanisms could not be more prevalent. Diffusion occurs throughout the human body, and without it, cells and body tissue could not get important nutrients for survival, the eyes would dry out, and many medicines could not be absorbed into the body. The lack of nutrients inside the cell, and between the cell and the blood vessel, creates a concentration gradient between the blood vessel and the cell. Due to the lower concentration in the cell, the nutrient diffuses through the blood vessel wall into the cell. The model of oxygen flow in the microcirculation by Pierce (2006) and Waniewski et al (1999) highlighted the inefficiency of diffusion as a means of oxygen transport in the blood and hence the use of microcirculation by the body in the form of oxygen bounded to the hemoglobin. Ibrahim, et al (2006) was able to present his model as a simple first order differential equation which shows that the amount of drug in the blood stream approaches a constant value under steady state conditions at a particular infusion rate. Burns and Donald (2004) developed a deterministic model for predicting drug molecule diffusion across the Blood-Brain Barrier (BBB).   Kool (2006), developed a model for antibiotic Distribution and Eradication of Bacteria causing endocartis based on predicting the duration of treatment of a patient suffering from endocartis focusing on the eradication or killing of the bacteria population. For medicines taken orally as pills, the medicine must somehow find its way into the bloodstream. If the pills capsule is a time release mechanism, the medicine must first diffuse out of the capsule into the stomach and then into the bloodstream, all by diffusion process Chaplya and Chernukha (2010), Pierce (2006) and Crank (1970). Another area of pharmacokinetic where diffusion plays an important role is in the release of drugs from a swellable collagen matrix. Works in this area have been investigated by Mario (2005), and Bause, et al (2006) among others.

From the foregoing, it is obvious that diffusion is the basic way in which all nutrients, gases, wastes and neurotransmitters move in the body (unless active transport is required) and involves the blood stream in one way or the other Bech (2008). Rao, P.T. and co, developed models for

blood glucose level in a type-2 Diabetes Mellitus patient, and also determined the optimal control policies of glucose regulatory system based on certain assumptions, Rao, et al (2011).

The importance of the knowledge of the operation of biological processes is made available by first building a model to represent such process from which the effects of the various factors that affects the processes are then determined, Vakalis (2002), VanLöbensels et al (1999) Bender (1978), Ottesen et al (2006) and Rice and Do (1995).

2.0 The Models.

Assumptions

Within a particular blood vessel, the speed of blood flow is taken as constant.

The radius of the blood vessels is also constant.

There exists a mechanism that instantaneously removes the diffused substrate from the tissue fluid into the cells thereby maintaining C_s constant.

Diffusion alone is able to account for the differences in the concentration of the substrate in the entrance and exit of the section under consideration.

The blood vessel is assumed to be of negligible thickness and its exterior is immersed in tissue and cell fluid.

The blood vessel will be considered as a long cylindrical pipe of length L in which there is a lateral diffusion as the blood flows with constant velocity through it.
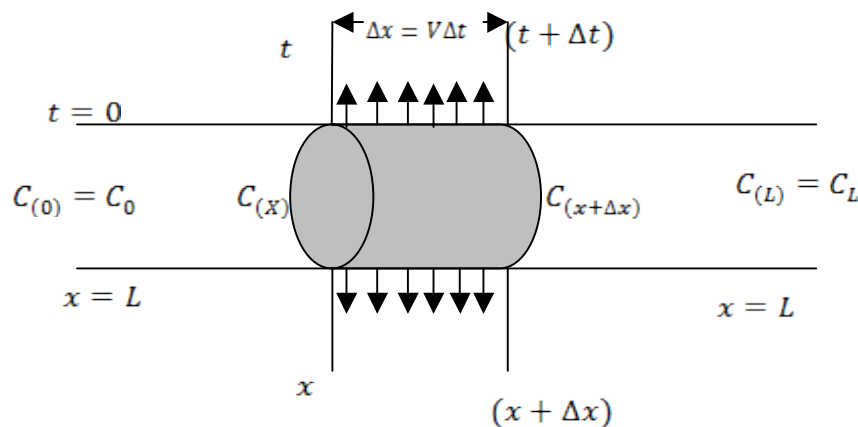

Fig. 1: Sketch of the cross section of a blood vessel

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For the derivation, we assume that the molecules of the solute move along the x-axis and denote the concentration at time t by $C_{(t)}$.

$C_{(x)}$ is the concentration of solute of interest at a position in the blood vessel;

C_s is the solute concentration in the surrounding tissue fluid;

V is the speed of blood flow;

D is the diffusivity constant of the solute in the blood;

r is the radius of the blood vessel.

$$V = \pi r^2 \Delta x \quad\text{...(1)}$$

$Amount\ at\ x = Concentration \times Volume\ at\ x$

$$= C_{(x)} \times \pi r^2 \Delta x \quad\text{.. (2)}$$

$Amount\ at\ x + \Delta x = Concentration \times Volume\ at\ x + \Delta x.$

$$= C_{(x+\Delta x)} \times \pi r^2 \Delta x\text{.........................(3).}$$

Amount of materials that have diffused over time is given as

$Rate\ of\ diffusion \times change\ in\ time.$

$$Rate\ of\ diffusion = D \times 2\pi r \Delta x \left[C_{(x+\theta\Delta x)} - C_s \right] \quad\text{.................................(4)}$$

where θ is such that $0 \le \theta \le 1$ is a typical point between x and $x + \Delta x$ at which to assume an average value of the concentration.

∴ Amount of materials that have diffused over time

$$= 2D\pi r \Delta x \left[C_{(x+\theta\Delta x)} - C_s \right] \Delta t \quad\text{........................... (5)}$$

Taking a material balance across the section shown:
$Accumulation = Input - Output + Generation - Consumption$ .... (6)
This is given as

$$\left[C_{(x)} - C_{(x+\Delta x)} \right] \pi r^2 \Delta x = 2D\pi r \Delta x \left[C_{(x+\theta\Delta x)} - C_s \right] \Delta t \quad\text{.................... (7)}$$

Developing the Model in terms of position

But $\quad \Delta x = V \Delta t$ .... (8)

Substitute into the left hand side of equation (7) to get

$$\left[C_{(x)} - C_{(x+\Delta x)} \right] \pi r^2 V \Delta t = 2D\pi r \Delta x \left[C_{(x+\theta\Delta x)} - C_s \right] \Delta t$$

Dividing through by Δx and rearranging, we have

$$\frac{\left[C_{(x)} - C_{(x+\Delta x)} \right]}{\Delta x} = \frac{2D}{rV} \left[C_{(x+\theta\Delta x)} - C_s \right].$$

$$\frac{[C_{(x+\Delta x)}-C_{(x)}]}{\Delta x}=-\frac{2D}{rV}\left[C_{(x+\theta\Delta x)}-C_s\right].$$

Taking limits as $\Delta x \to 0$

$$\lim_{\Delta x \to 0}\frac{[C_{(x+\Delta x)}-C_{(x)}]}{\Delta x}=-\frac{2D}{rV}\lim_{\Delta x \to 0}\left[C_{(x+\theta\Delta x)}-C_s\right].$$

$$\frac{dC_{(x)}}{dx}=-\frac{2D}{rV}\left[C_{(x)}-C_s\right]\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(9)$$

Equation (9) gives the required model in terms of the position under consideration.

Developing the Model in terms of time

To develop the model with respect to time, the mass balance is taken with respect to time and divide equation (7) through by Δt and taking limits as $\Delta t \to 0$, we have

$$\frac{dC_{(t)}}{dt}=-\frac{2D}{r}\left[C_{(t)}-C_s\right]\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(10)$$

Equations (9) and (10) are the two models which describes the system under consideration independently with respect to position and time respectively.

3.0 Solution to the Models.

 The models were solved both analytically and numerically and their solutions compared.

The model for the position effect is given as

$$\frac{dC_{(x)}}{dx}=-\frac{2D}{rV}\left[C_{(x)}-C_s\right]\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(9)$$

Rearranging equation (9)

$$\frac{dC_{(x)}}{[C_{(x)}-C_s]}=-\frac{2D}{rV}dx\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(11)$$

Integrating, we have

$$\int_{C_o}^{C_{(x)}}\frac{dC_{(x)}}{[C_{(x)}-C_s]}=-\frac{2D}{rV}\int_{x_o=0}^{x}dx.$$

$$\ln[C_{(x)}-C_s]\Big|_{C_o}^{C_{(x)}}=-\frac{2D}{rV}x\Big|_o^x.$$

$$\ln[C_{(x)}-C_s]-\ln[C_0-C_s]=-\frac{2Dx}{rV}$$

$$\ln\left[\frac{C_{(x)}-C_s}{C_0-C_s}\right]=-\frac{2Dx}{rV}\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(12)$$

$$\frac{C_{(x)} - C_s}{C_0 - C_s} = \ell^{-\frac{2Dx}{rV}}$$ .... (13)

$$C_{(x)} = C_s + [C_0 - C_s]e^{-\frac{2Dx}{rV}}$$ .... (13b)

If we know only C_s, C_0, C_L and L; by substituting $x = L$ and $C = C_L$ into equation (12), we have

$$\ln\left[\frac{C_L - C_s}{C_0 - C_s}\right] = -\frac{2DL}{rV}$$ ....(14)

By dividing equation (12) by equation (14) yields $\quad \frac{\ln\left[\frac{C_{(x)} - C_s}{C_0 - C_s}\right]}{\ln\left[\frac{C_L - C_s}{C_0 - C_s}\right]} = \frac{x}{L}$

$$\ln\left[\frac{C_{(x)} - C_s}{C_0 - C_s}\right] = \frac{x}{L}\ln\left[\frac{C_L - C_s}{C_0 - C_s}\right]$$..(15)

$$\ln\left[\frac{C_{(x)} - C_s}{C_0 - C_s}\right] = \ln\left[\frac{C_L - C_s}{C_0 - C_s}\right]^{\frac{x}{L}}$$ ....(16)

And finally,

$$C_{(x)} = C_s + [C_0 - C_s]\left[\frac{C_L - C_s}{C_0 - C_s}\right]^{\frac{x}{L}}$$................................(17)

Therefore, in the absence of some of the data needed in equation (13b), our model can still be discussed to know how the concentration of our solute of interest varies along the blood vessel.

The model for the time effect is given as

$$\frac{dC_{(t)}}{dt} = -\frac{2D}{r}[C_{(t)} - C_s]$$..(10)

where $C_{(t)}$ is the concentration of our desired solute at time t in a particular position in the blood vessel.

Other parameters remain as defined previously.

Similarly, rearranging equation (10) and solving, we have

$$C_{(t)} = C_s + [C_0 - C_s]\ell^{-\frac{2Dt}{r}}$$ .... (18)

The same transformation were performed when we know only C_s, C_0, C_τ and τ to finally get

$$C_{(t)} = C_s + [C_0 - C_s]\left[\frac{C_\tau - C_s}{C_0 - C_s}\right]^{\frac{t}{\tau}}$$ .... (19)

4.0 Results and Discussion of Results

The initial data used in the numerical solution and the simulation of the system is as follows:
$C_S=0.005$; $C_O=0.55$; $C_L=0.005$; $L=60$; $D=2.35 \times 10^{-5}$; $V=1.0$; $r=5 \times 10^{-5}$
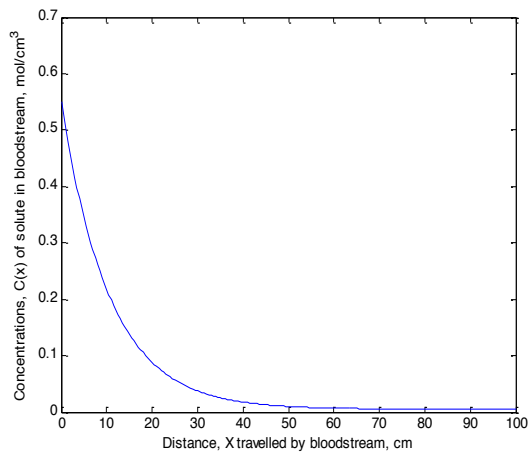$X_O = 0.00$;

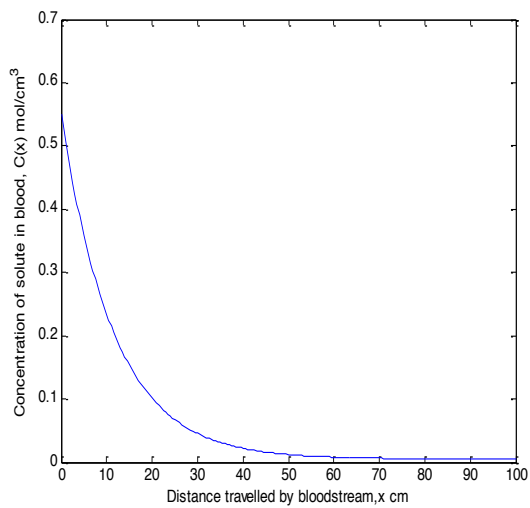Figure.2: Graph of Concentration of solute C(x) against distance, x

Figure.3: Graph of Concentration of solute C(t) against time, t

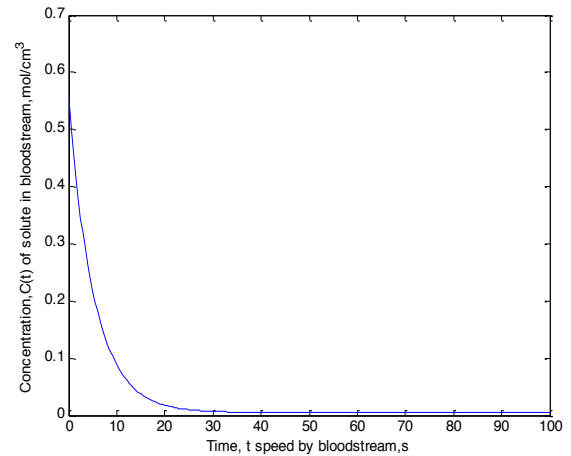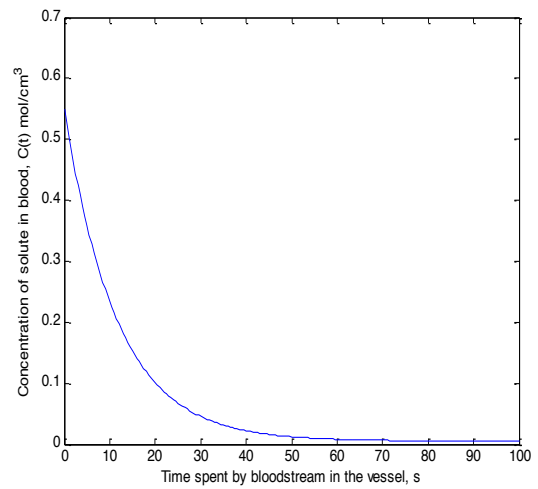Figure.4: Graph of Concentration of solute C (x) against distance, x without D,V,R.

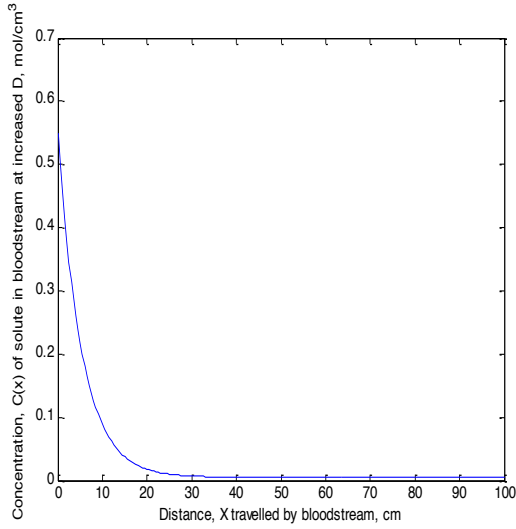Figure.5: Graph of Concentration of solute in blood against time, t without D,V,R

Fgure.6: Graph of Concentration of solute
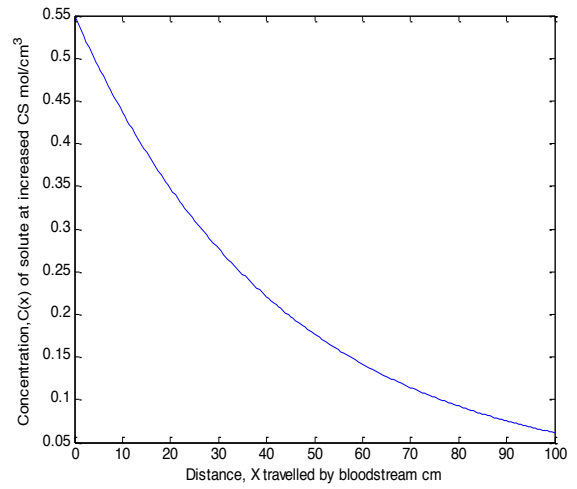C(x) at increased D against distance, X

Figure.8: Graph of Concentration of solute
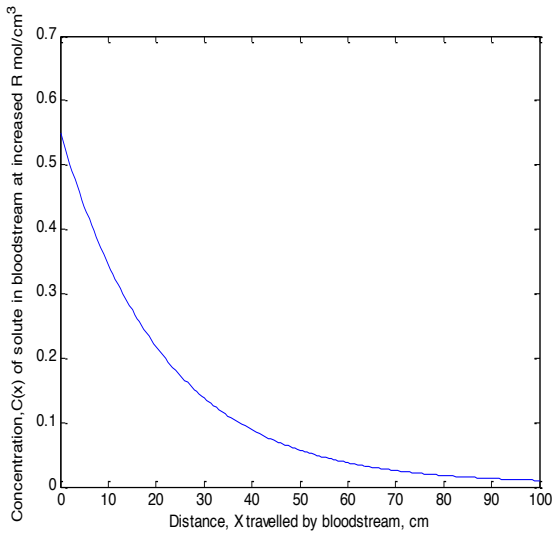C(x) at increased C_S against distance, x

Figure.7: Graph of Concentration of solute
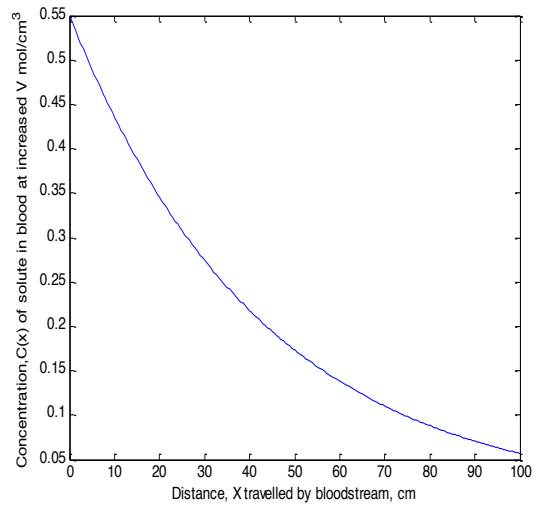C(x) at increased R against distance, x

Figure.9: Graph of Concentration of solute
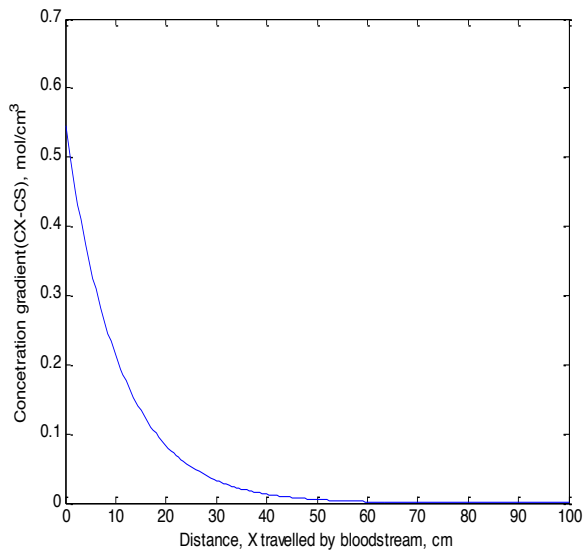C(x) at increased V against distance, x

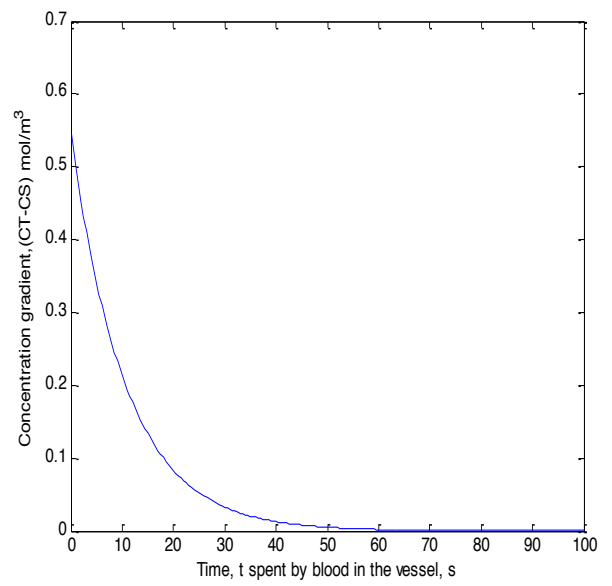Figure.10: Graph of Concentration gradient $(C_X - C_S)$ against distance, x

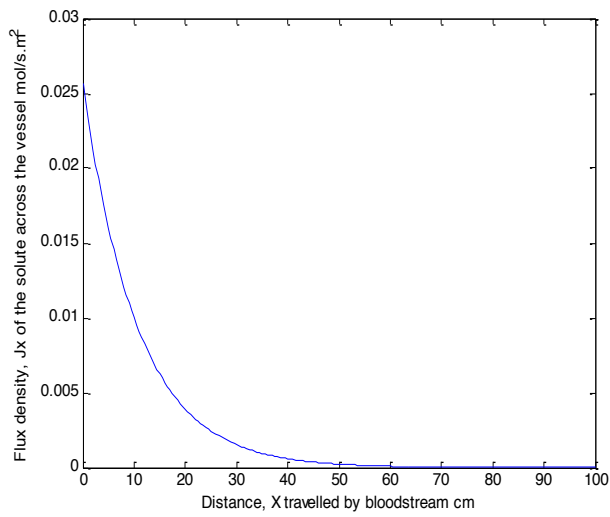Figure.11: Graph of Concentration gradient, $(C_T - C_S)$ against time, t

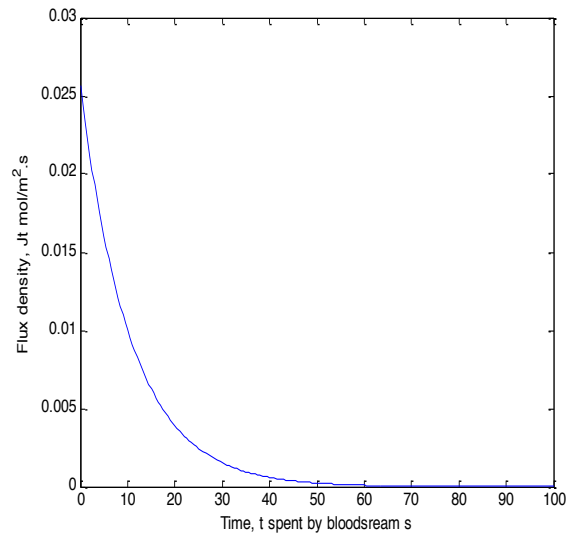Figure.12: Graph of Flux density of solute, J_x against distance, x

Figure.13: Graph of Flux density, J_t against time, t

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Equations (13b) and (18) shows that the concentration profile of a diffusing substance in the bloodstream is a function of the diameter (or radius) of the blood vessel, speed of blood flow, diffusivity of the substance in the blood, concentration gradient between the solute of the substance in the blood and that in the surrounding tissue, the time spent and position of the blood fluid in the body. We can therefore determine the exact concentration of the solute at any position in the blood vessel and hence monitor its movement in the body.

It was noticed from the curves generated that the concentration, C_x decreases exponentially with an increase in distance travelled by the bloodstream as well as time spent in the blood vessel, Figure 2 and 3. As the distance travelled and time spent tends to infinity, the concentration gradient decreases and the concentration of the solute in the bloodstream approach the concentration of the solute in the surrounding tissue C_s. At this point equilibrium is said to have been reached and the net diffusion is zero. In the absence of diffusivity constant D, blood speed V and radius R of blood vessel values, we were able to study the solute concentration profile with respect to distance and time, knowing the initial concentration of solute in blood, concentration of solute in the tissue, length of the blood vessel and the concentration at the end of the blood vessel using equations (17) and (19) respectively.

It was also seen that in the absence of diffusivity constant D, blood speed V and the radius R of blood vessel, the concentration, C_x decreases exponentially with an increase in distance travelled by the bloodstream as well as time spent in the blood vessel, Figures 4 and 5.

An increase in diffusivity constant D enhances diffusion and reduces the concentration of the solute in the bloodstream much faster hence resulted to the concentration of the solute in the bloodstream reaching the steady state value at both lesser distance and time Figure 6.

Also an increase in radius of the blood vessel R, the velocity of the bloodstream V, and the concentration of the solute in the surrounding tissue C_S, reduces the concentration gradient and the rate of diffusion and as such the concentration of the solute in the bloodstream C_X, may not even get to the concentration of solute in the surrounding tissue, C_S at the end of blood vessel L Figures 7, 8 and 9. It was again noticed that the concentration gradient decays exponentially with distance travelled as well as the time spent by the blood fluids Figures 10 and 11.

It was also noticed that the flux density decreases with distance travelled and the time spent by the blood fluid Figures 12 and 13. It was observed from the equations and the curves that large

blood vessel beyond (5–10μm in diameter) makes the exponential function to approach zero implying that relatively no diffusion has taken place at such diameter or radius. The computer codes, MATLAB programs and numerical techniques of solving the models using MATLAB Euler's method used in solving the models and generating the graphs for the models were presented in the appendices.

5.0 Conclusions.

The solution to the models reveals that the concentration gradient decays exponentially with distance and time until an equilibrium is reached. The flux density decreases with an increase in distance travelled by bloodstream x, time spent by bloodstream t, and reduction in concentration gradient (C_X-C_S), and approach zero as concentration gradient approach zero, that is concentration of solute in bloodstream becomes equal to that in the surrounding tissue.

Hence the flux density depends majorly on the concentration gradient between the solute in the bloodstream and that in the surrounding fluid, the distance travelled by the blood solution in the blood vessel, time spent and the velocity of the blood solution. The usefulness of this work have been identified to include but not limited to nutrient uptake from the blood, infections by pathogenic secretions, dialysis, drug action, gaseous exchange etc. These models however were limited due to the assumption that the solute concentration in the tissue is constant by assuming a mechanism that quickly removes the diffused solute from the tissue fluid, which does not always apply.

Finally, within the limits of acceptability, the models can account for many diffusion processes in the human blood stream.

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