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Compressive Sensing Radar based on Random Chaos

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Abstract

In complicated electromagnetic environments, the "Four-Anti" capabilities of radar and highly efficient signal processing approaches are crucial. A new radar system with a randomly transmitted signal and a compressive sensing imaging method based on random chaotic sequence (RCS)are proposed. The randomness and statistical independence of the radar waveform are proved. Combined with the imaging model ofradar, a random chaotic sensing matrix (RCSM) is presented and proved to satisfy the restricted isometry property (RIP) with overwhelming probability. Numerical simulations comparison with other random matrices are conducted and demonstrate that the performance of these sensing matrices are almost the same. However, the RCSM can be easily implemented in hardware and is more suitable for scenarioswhererequire security and strong anti-jamming ability.

Key words

Compressive sensing, sensing matrix, random chaos, restricted isometry property, radar imaging

1. Introduction

Compressive sensing (CS) has become an increasingly important research field in applied mathematics, computer science, and electrical engineering [1-5]. One of the fundamental tasks in CS is to design a sensing matrix that is sufficient to ensure the exact recovery of the original signal frommuch fewer measurements. Many properties and characteristics of sensing matrixare presented to ensure a unique and stable signal reconstruction [6-8]. Depending on the construction method, sensing matrices can be divided into two categories: the deterministic matrix and the random matrix, the latter has been certified to satisfy RIP with high probability. Unfortunately, the fully random matrixis sometimes impractical to implement in hardware.

Chaos is a kind of phenomena arises in deterministic nonlinear dynamical system, which is similar to random signal. This property motivates us to employ a chaotic system in CS. Linh-Trung et al. [9] adopted a chaos filter to construct the sensing matrix. Yu et al. [10] proposed constructing the sensing matrix with a chaotic sequence and demonstrated that this type of sensing matrix satisfies the RIP with overwhelming probability. Felix Krahmer et al. [11] presented a bound for the suprema of chaotic processes, which improved the estimates for the RIP of structured random matrices. Kafedziski et al. [12] demonstrated that Chua and Lorenz chaotic sequences are suitable for sensing matrix. However, the intrinsic determinacy of such systems determines the internal structure and determinacy of chaotic sequences, and, strictly speaking, no truly random sensing matrix is available by these approaches.

In view of such cases, we propose to employ a random chaotic sequence (RCS) to generate a sensing matrix. First, we evaluate the randomness and statistical independence of the RCS and use it to construct the random chaotic sensing matrix (RCSM) for CS. Then, the RCSM is shown to satisfy the RIP with overwhelming probability. Several numerical simulations are presented to demonstrate the efficiency of the RCSM. The RCSM has a similar performance to that of the common sensing matrices.

2. The fundamental theory of CS

Let $x_s \in R^N$ be a discrete signal, given an orthonormal basis matrix or a frame $\Psi \in R^{N\times N}$, and x_s can be represented in terms of Ψ as

$$x_s = \Psi\alpha \tag{1}$$

With only $k \ll N$ nonzero entries in $\alpha \in R^N$, we call x_s a k-sparse signal under Ψ. Compressive sensing can be viewed as a linear measurement

$$y_s = \Phi \cdot x_s + n \tag{2}$$

where $\Phi \in R^{M\times N}$ is called the sensing matrix, $y_s \in R^M$ ($M < N$)is the measurement vector, and n represents additive noise. By combining (1) and (2) we obtain

$$y_s = \Phi \cdot x_s + n = \Phi\Psi\alpha + n = \Theta\alpha + n \tag{3}$$

where $\Theta = \Phi\Psi$ is an $M \times N$ matrix [13]. When Θ satisfies theRIP, α can be recovered from y_s with overwhelming probability by solving

$$\min \|\alpha\|_{l_1} \quad \text{s.t.} \quad \|\Theta\alpha - y_s\|_{l_2} \le \varepsilon \tag{4}$$

where $\|\square\|_{l_1}$ denotes the l_1-norm, ε is an upper bound on the size of the noise contribution, and $\|n\|_{l_2} \le \varepsilon$.

3. Random chaotic sequence and its statistical characteristics

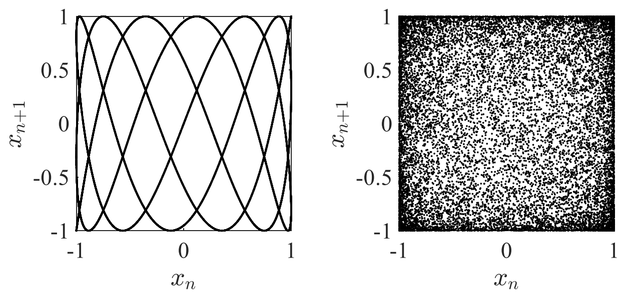
3.1 Random chaotic sequence

$x_n = \sin^2\left(\theta\pi 2^n\right)$ has been proved to be the exact solution for $x_{n+1} = 4x_n\left(1 - x_n\right)$, which is a well-known Logistic map [14-17]. The exact general solution to many other maps can be expressed as $x_n = P\left(\theta T k^n\right)$, where $P\left(t\right)$ is a periodic function with the period T, θ is a real number, and k is an integer. In this paper, we investigatethe randomness and statistical independence of the sequence

$$x_n = \cos\left(2\pi\theta z^n\right) \tag{5}$$

where $1 < z \in R$.

Suppose that $z = p/q$,where p and q are relatively prime integers. It can be proved that given the sequence $x_0, x_1, \ldots, x_m$ generated by Eq.(5), the value of the next point x_{m+1} is uncertain because it can take q different values.Fig.1 shows the first-return map of sequences generated by Eq. (5) with different values of z .



(a) (b)

Fig. 1. First-return map produced by Eq.(5). (a) $z = 13/5$; (b) $z = \pi$

Let us define the sequences parameterized by integer k as follows:

$$x_n^{(k,m)} = \cos\left[2\pi\left(\theta_0 + q^m k\right)\left(\frac{p}{q}\right)^n\right] \tag{6}$$

We have $x_0 = x_1 = \cdots = x_n$ for all $n \le m$ because

$$x_n^{(k,m)} = \cos\left[2\pi\theta_0\left(\frac{p}{q}\right)^n + 2\pi k p^n q^{m-n}\right] = \cos\left[2\pi\theta_0\left(\frac{p}{q}\right)^n\right] \tag{7}$$

However, the next value

$$x_{m+1}^{(k,m)} = \cos\left[2\pi\theta_0\left(\frac{p}{q}\right)^{m+1} + 2\pi k\frac{p^{(m+1)}}{q}\right] \tag{8}$$

can take q different values. Thus, $x_n^{(k,m)}$ is forward-unpredictable.It can be proved that $x_n^{(k,m)}$ is also backward-unpredictable by defining the sequence families

$$x_n^{(k,m,s)} = \cos\left[2\pi\left(\theta_0 + q^m k\right)\left(\frac{q}{p}\right)^s\left(\frac{p}{q}\right)^n\right] \tag{9}$$

Given the sequence $x_s, x_{s+1}, \ldots, x_{s+m}$, x_{s-1} is unpredictable because

$$x_{s-1}^{(k,m,s)} = \cos\left[2\pi\left(\theta_0 + q^m k\right)\left(\frac{q}{p}\right)^s\left(\frac{p}{q}\right)^{s-1}\right] = \cos\left(2\pi\theta_0\frac{q}{p} + 2\pi k\frac{q^{m+1}}{p}\right) \tag{10}$$

Consequently, x_{s-1} has p different possible values. When z is an irrational number, the future and past points can take infinite possible values. Therefore, such a sequence is truly random.

3.2 Statistical independence

x_n generated by Eq. (5) has the invariant density $\rho(x) = 1/\left(\pi\sqrt{1-x^2}\right)$. The m-th moment of x_n satisfies $E\left(x_n^m\right) = 0$ if m is odd and

$$E\left(x_n^m\right) = 2^{-m}\binom{m}{\frac{m}{2}} \tag{11}$$

if m is even.

It has been proved that the sequence generated by Eq. (5) is statistically independent for a transcendental value of z, but this is not true for algebraic numbers [18]. However, the high-order correlations with sampling distance can be used to measure the independence. We have the following lemma.

Lemma 1*Suppose that* $X = \{x_n, x_{n+1}, \ldots, x_{n+k}, \ldots\}$ *is the sequence generated by* (5), *given the sampling* distance d *, for any integer* $0 < m_0, m_1 < z^d$ *, we have*

$$E\left(x_n^{m_0} x_{n+d}^{m_1}\right)=E\left(x_n^{m_0}\right)E\left(x_{n+d}^{m_1}\right) \tag{12}$$

Proof: If m_i is odd, the right-hand side of (12) is equal to 0. For the left-hand side, considering that $\cos\theta=\left(e^{j\theta}+e^{j\theta}\right)/2$, we have

$$\begin{aligned}E\left(x_n^{m_0} x_{n+d}^{m_1}\right)&=\int_{-1}^{1}\rho\left(x_0\right)x_n^{m_0}x_{n+d}^{m_1}dx_0\\&=\int_0^1\cos^{m_0}\left(2\pi\theta z^n\right)\cos^{m_1}\left(2\pi\theta z^{n+d}\right)d\theta\\&=z^{-(m_0+m_1)}\sum_{\sigma}\delta\left[z^n\sum_{i=1}^{m_0}\sigma_{n_i}+z^{n+d}\sum_{i=1}^{m_1}\sigma_{(n+d)_i}\right]\end{aligned} \tag{13}$$

The last equation uses the fact that $\int_0^1 e^{j2\pi\theta k}d\theta=\delta\left(k\right)$, with $\delta\left(k\right)=0$ if $k\neq 0$ and 1 otherwise. Σ_{σ} represents summation over all possible combinations with $\sigma_i=\pm 1$. All possible cases are analyzed below:

(i) Both m_0 and m_1 are odd: $\left|\Sigma_{i=1}^{m_0}\sigma_{n_i}\right|\leq m_0$ and $\left|\Sigma_{i=1}^{m_1}\sigma_{(n+d)_i}\right|\geq 1$. Hence, $z^n\Sigma_{i=1}^{m_0}\sigma_{n_i}+z^{n+d}\Sigma_{i=1}^{m_1}\sigma_{(n+d)_i}\neq 0$;

(ii) Either m_0 or m_1 is odd and the other is even. Without loss of generality, suppose that m_0 is odd and m_1 even. Under this scenario, it is possible that $\Sigma_{i=1}^{m_1}\sigma_{(n+d)_i}=0$, whereas $\Sigma_{i=1}^{m_0}\sigma_{n_i}\neq 0$,hence, $z^n\sum_{i=1}^{m_0}\sigma_{n_i}+\sum_{i=1}^{m_1}\sigma_{(n+d)_i}\neq 0$.

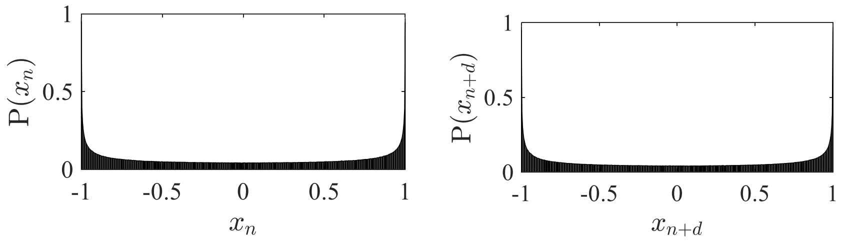
Then, the left-hand side of (12) is also equal to 0.

(iii) Both m_0 and m_1 are even numbers,after a common combinatorial calculation, we obtain

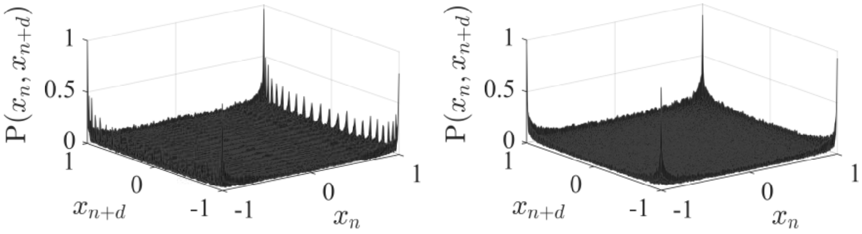
$$E\left(x_n^{m_0} x_{n+d}^{m_1}\right)=2^{-(m_0+m_1)}\binom{m_0}{m_0/2}\binom{m_1}{m_1/2} \tag{14}$$

Comparing equation (14) with equation (11), we obtain (12).

Lemma 1 implies that x_n and x_{n+d} are statisticallyindependent when the sampling distance $d\rightarrow\infty$. If d is sufficiently large, for instance, $d=10$ with $z=13/5$, we have $E\left(x_n^{m_0}x_{n+d}^{m_1}\right)=E\left(x_n^{m_0}\right)E\left(x_{n+d}^{m_1}\right)$ for all $m_0,m_1<14117$,then, x_n and x_{n+d} can be considered approximately independent, as illustrated in Fig.2.



(a) (b)



(c) (d)

Fig. 2. Probability density. (a) $P(x_n)$ and (b) $P(x_{n+d})$; (c) and (d) joint probability density $P(x_n, x_{n+d})$ for sampling distance $d = 5, 10$

4. Random chaotic sensing matrix for the radar system

Let $x[k] = \{x_n, x_{n+d}, \ldots, x_{n+kd}\}$ be the chaotic sequence extracted from the sequence generated by (5) with $z = 13/5, d = 10$, which is used as the transmitted radar signal and possessesthe characteristics of being fully random and statistically independent.

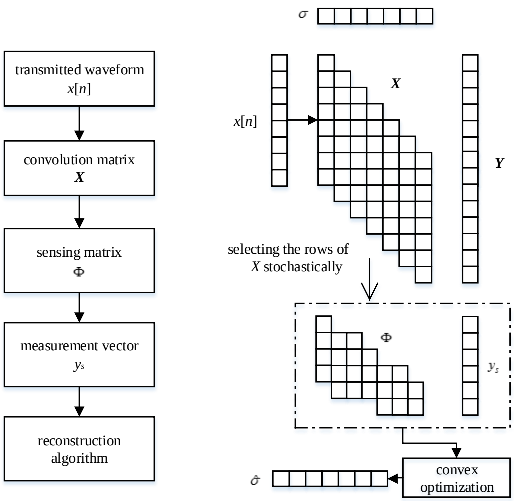
According to the principles of radar imaging, it can be assumed that the returned signal $y(t)$ can be modeled as the convolution of the transmitted waveform $x(t)$ with the reflectivity of the observed scene $\sigma(t)$ [19]:

$$y(t) = x(t) * \sigma(t) = \int_{-\infty}^{\infty} x(t-\tau)\sigma(\tau)d\tau \tag{15}$$

The corresponding discrete version can be written in matrix form as

$$\begin{bmatrix} y[1] \\ y[2] \\ \vdots \\ y[N] \\ y[N+1] \\ \vdots \\ y[2N-2] \\ y[2N-1] \end{bmatrix} = \begin{bmatrix} x[1] & 0 & \cdots & 0 \\ x[2] & x[1] & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ x[N] & x[N-1] & \cdots & x[1] \\ 0 & x[N] & \cdots & x[2] \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & x[N-1] \\ 0 & 0 & \cdots & x[N] \end{bmatrix} \square \begin{bmatrix} \sigma[1] \\ \sigma[2] \\ \vdots \\ \sigma[N-1] \\ \sigma[N] \end{bmatrix} \tag{15}$$

or more compactly as $Y = X \cdot \Sigma$. Compared with Eq. (3), formally speaking, CS is well-suited for the radar system. Our interest is how to design the sensing matrix Φ with the RIP based on X. The algorithm implementing procedure of CS radaris shown in Fig.3.



(a) (b)

Fig.3. Algorithm implementing procedure of CS radar.

(a) algorithm flowchart; (b) algorithm diagram

Let $D_M : R^N \rightarrow R^M$ be an operatorthat restricts a vectorto its entries in T and $T \subset \{1, 2, \ldots, N\}$ be a set of cardinality M. We construct the random chaotic sensing matrix $\Phi \in R^{M \times N}$ by utilizing the matrix X in the following manner:

$$\Phi = D_M X$$

(16)

Theorem 1 *The RCSM* Φ *in Eq.* (17) *satisfies the RIP with probability* $\Pr \geq 1 - 2\exp\left(-C_1 \cdot M\right)$ *for any* $M \geq \left[C_2 \cdot k \cdot \log\left(N/k\right)\right]$ *with* $\delta_k \in (0,1)$, *where* C_1, C_2 *depends only on* δ_k.

To prove Theorem 1, we introduce Lemma 2. Let $\Omega \subseteq \{1, 2, \ldots, N\}$ denote the set column indices and Φ_Ω be an $M \times |\Omega|$ sub-matrix of Φ with indices Ω.

Lemma 2 *Given a matrix* $\Phi \in R^{M \times N}$ *whose entries extracted independently from a certain distribution, for any set of indices* Ω *with* $|\Omega| = k < M$, $\exists\, \delta \in (0, 0.2)$, *such that*

$$\forall w \in R^{|\Omega|}, (1 - 5\delta)\|w\|_2^2 \leq \|\Phi_\Omega \cdot w\|_2^2 \leq (1 + 5\delta)\|w\|_2^2$$

(17)

withhigh probability

$$\Pr \geq 1 - 2\left(\frac{12}{5\delta}\right)^k \cdot \exp\left(-c(\delta) M\right) \tag{18}$$

where $R^{|\Omega|}$ *denotes the set of all vectors in* R^N *that are zero outside of* T *.*

Using Lemma 2, it is easy to prove Theorem 1.

According to Lemma 2, the RCSM will fail to satisfy Eq. (18) with probability

$$\Pr \le 2\left(\frac{12}{\delta_k}\right)^k \cdot \exp\left(-c\left(\frac{\delta_k}{5}\right)M\right) \tag{19}$$

for each R^k, where $\delta_k \in (0,1)$. There are $\binom{N}{k}$ $R^k \subset R^N$,thus,the RCSM will violate the RIP with the small probability

$$\begin{aligned}\Pr &\le 2\binom{N}{k}\cdot\left(\frac{12}{\delta_k}\right)^k \cdot \exp\left(-c\left(\frac{\delta_k}{5}\right)M\right)\\ &\le 2\cdot\exp\left[-c\left(\frac{\delta_k}{5}\right)M + k\left(\log\frac{eN}{k}+\log\frac{12}{\delta_k}\right)\right]\\ &= 2e^Q\end{aligned} \tag{20}$$

Subsequently, for a fixed $C_1 > 0$, whenever $k \le \left[(C_1 \cdot M)/\log(N/k)\right]$, we obtain $Q \le -C_2 \cdot M$ if $C_2 \le c(\delta_k/5) - C_1\left[1+\left(1+\log(12/\delta_k)\right)/\log(N/k)\right]$.

5. Numerical experiments

The scattering characteristics of a radar target can be represented by several scattering centers in high frequency. To verify the validity of the proposed compressive sensing radar imaging system in this paper, a plot of a radar scene with 52 scatters with different reflectivities is shown in Fig. 4, and thecorresponding vectorized version is shown inFig. 5. The sparsity of the signal is $k = 52$. Considering the effect of thermal noise in the radar receiver, we add Gaussian noise to the initial signal with a signal-to-noise ratio SNR=40. The radar parameters are shown in table 1.

Table 1. The radar parameters

carrier frequency	bandwith	pulse duration	sampling frequency
2 GHz	250 MHz	2.5 μs	1GHz

The scene vector is sampled by the RCSM with dimensions of 500×2500, i.e., a sampling rate of 0.2. y_s is the measurement vectorshown in Fig. 6. Subsequently, we reconstruct the signal x_s^* via convex optimization technique illustrated in Fig. 7, and the recovered radar scene is compared with the original scene shown in Fig. 8. The corresponding recovery error is depicted in Fig. 9. Fig. 10 is the two-dimensional recovered radar scene. For comparison purposes, the RCSM, Gaussian sensing matrix (GSM) [20], Bernoulli sensing matrix (BSM) [21], and sparse random sensing matrix (SRSM) [22] are used to sample the same scene vector with different measurement number M. The absolute error of recovery and the corresponding SNR are illustrated in Fig. 11 and Fig. 12, respectively. With the same measurement number $M = 800$, the RCSM , GSM, BSM and SRSM are adopted to measure the scene vector with variational sparsity k, and the probability of successful recovery as a function of k is depicted in Fig. 13. The criterion of successful reconstruction is $\|x_s - x_s^*\| \le 0.002$.

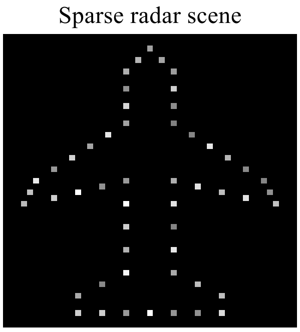


Fig. 4. Sparse radar scene

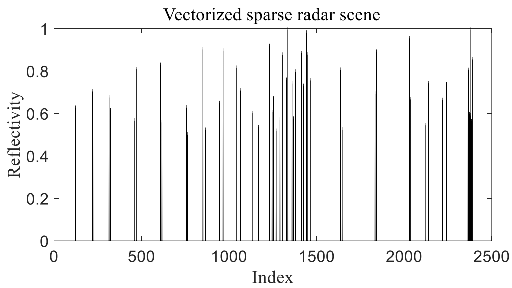


Fig. 5. Vectorized sparse radar scene

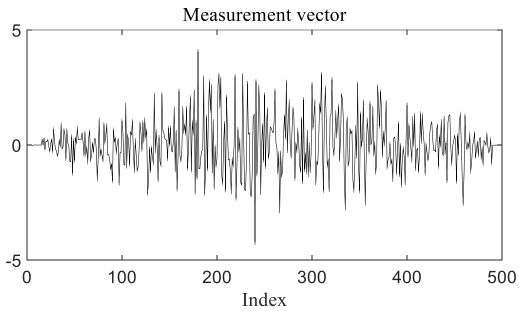


Fig. 6. Measurement vector

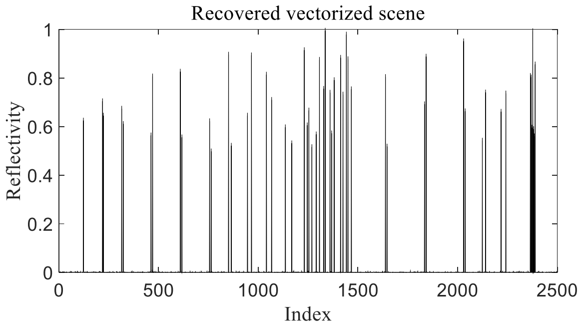


Fig. 7. Recovered vectorized radar scene

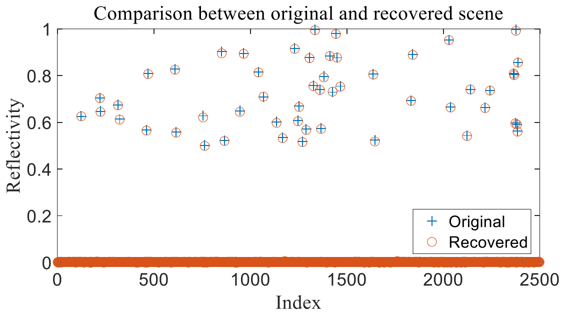


Fig. 8. Recovered scene compared with the original radar scene

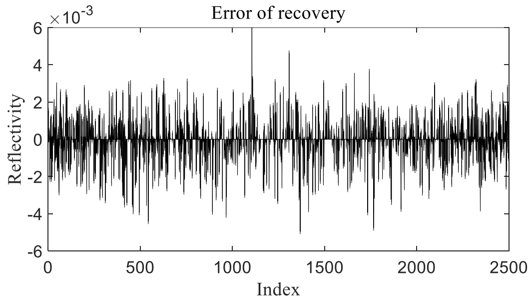


Fig. 9. Error of recovery

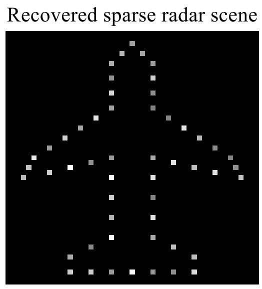


Fig. 10. Recovered sparse radar scene

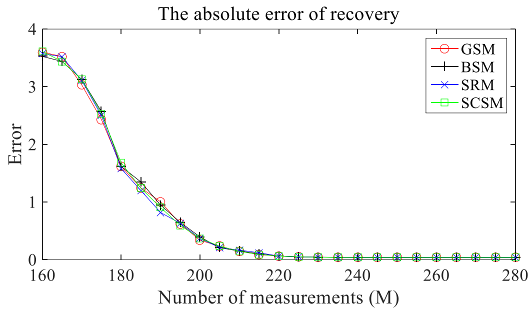


Fig. 11. Absolute error of recovery for different sensing matrices

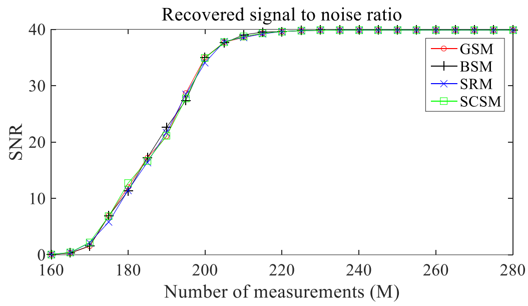


Fig. 12. SNRs for different sensing matrices

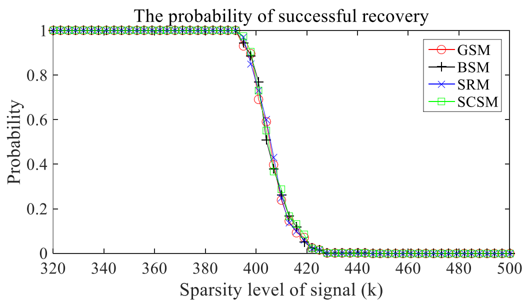


Fig. 13. Probability of successful recovery for different sensing matrices

6. Discussion

The RCS is an ideal radar waveform for its strong anti-interference ability, low probability of intercept, good electromagnetic compatibility. In contrast to random noise, the generation, replication and control process of the RCS are very easy.

The CS radar offers a framework for the detection of sparse signals for radar with a high-resolution. As we known, sensing matrix plays a key role in CS, currently, two kinds of problems exist in the design of sensing matrix. One is that the random sensing matrix is difficult to be realized by hardware. The other is that the size of deterministic matrix cannot be arbitrary. To cope with these problems, a novel algorithm is proposed based on CS for random chaotic radar system. Projection for low dimension data is adopted instead of correlation; Signal recovery used to substitute pulse compression. In this algorithm, detected targets in scene satisfy the requirement of sparsity peculiarity, and sensing matrix is constructed by selecting the rows of convolution matrix stochastically. Furthermore, convex optimization method is applied to reconstruct target signals, and reconstruction error is significantly reduced and sidelobes are faithfully suppressed.

Conclusions

Connecting with the structure features of the radar imaging system, a new radar system and a sensing matrix based on a random chaotic sequence, which is fully random and statistically independent, are proposed. The RCSM is proved to satisfy the RIP with overwhelming probability, which guarantees exact recovery. Simulations of a sparse radar scene CS were performed, and a comparison among the RCSM, GSM, BSM and SRSM demonstrated that RCSM exhibits similar performance to the other methods, but the RCSM is easily implemented in hardware.

The number of measurement and sparsity of the original signal affected the performance of reconstruction, the error of recovery becomes greater with the decrease of measurement number M , and the probability of successful reconstruction gets smaller with the increase of signal sparsity k , therefore, all of these should be taken into account in a practical application.

In a word, as a new signal processing theory, CS provides great possibilities for overcoming inherent limitations of traditional radar, and has potential to resolve many problems associated with high resolution radar, such as high sampling rate, too many dada and difficulties of real time processing.

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Conflicts of Interest

The authors declare no conflict of interest.

References

1. Baraniuk, Richard G. "Compressive sensing", 2007, IEEE signal processing magazine, vol . 24, No.4, 118-121.
2. Candès, Emmanuel J., Justin Romberg, and Terence Tao. "Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information", 2006, IEEE Transactions on information theory, vol. 52, No. 2, 489-509.
3. Candes, Emmanuel J., and Terence Tao. "Decoding by linear programming", 2005, IEEE transactions on information theory, vol. 51, No. 12, 4203-4215.
4. Donoho, David L. "Compressed sensing", 2006, IEEE Transactions on information theory, vol. 52, No. 4, 1289-1306.
5. Malloy, Matthew L., and Robert D. Nowak. "Near-optimal adaptive compressed sensing", 2014, IEEE Transactions on Information Theory, vol. 60, No.7, 4001-4012.
6. Candès, Emmanuel J., and Michael B. Wakin. "An introduction to compressive sampling", 2008, IEEE signal processing magazine, vol. 25, No.2, 21-30.
7. Cahill, Jameson, Xuemei Chen, and Rongrong Wang. "The gap between the null space property and the restricted isometry property", 2016, Linear Algebra and its Applications, vol. 501, 363-375.
8. James, Oliver. "On the PDF of the square of constrained minimal singular value for robust signal recovery analysis", 2015, Coimbatore, India, February 2015, ICECS on. IEEE, 1701-1705.

9. Linh-Trung, Nguyen, Dinh Van Phong, Zahir M. Hussain, Huu Tue Huynh, Victoria L. Morgan, and John C. Gore. "Compressed sensing using chaos filters",2008, Adelaide, Australia, December 2008, ATNACon. IEEE, 219-223.
10. Yu, Lei, Jean-Pierre Barbot, Gang Zheng, and Hong Sun "Compressive sensing with chaotic sequence", 2010, IEEE Signal Processing Letters, vol. 17, No.8, 731-734.
11. Krahmer, Felix, Shahar Mendelson, and Holger Rauhut. "Suprema of chaos processes and the restricted isometry property", 2014, Communications on Pure and Applied Mathematics, vol. 67, No.11, 1877-1904.
12. Kafedziski, Venceslav, and Toni Stojanovski. "Compressive sampling with chaotic dynamical systems", 2011, Belgrade, Serbia, November 2011, TELFOR on. IEEE, 1877-1904.
13. Boche, Holger, Robert Calderbank, Gitta Kutyniok, and Jan Vybiral. Compressed Sensing and its Applications. 2015, Birkhäuser, Mathematics.
14. Kawamoto, Shunji, and Toshihide Tsubata. "Integrable chaos maps", 1996, Journal of the Physical Society of Japan, vol. 65, No.9, 3078-3079.
15. Kawamoto, Shunji, and Toshihide Tsubata. "The Weierstrass function of chaos map with exact solution", 1997, Journal of the Physical Society of Japan, vol. 66, No.8, 2209-2210.
16. Umeno, Ken. "Method of constructing exactly solvable chaos", 1997, Physical Review E, vol. 55, No.5, 5280-5291.
17. Kohda, Tohru, and Hirohi Fujisaki. "Jacobian elliptic Chebyshev rational maps", 2001, Physica D: Nonlinear Phenomena, vol. 148, No.3, 242-254.
18. Wang, Kai, Wenjiang Pei, Haishan Xia, Monica Garcia Nustes, and J. A. Gonzalez. "Statistical independence in nonlinear maps coupled to non-invertible transformations", 2008, Physics Letters A, vol. 372, No.44, 6593-6601.
19. Alonso, Mariví Tello, Paco López-Dekker, and Jordi J. Mallorquí. "A novel strategy for radar imaging based on compressive sensing", 2010, IEEE Transactions on Geoscience and Remote Sensing, vol. 48, No.12, 4285-4295.
20. Zhong, Kai, Prateek Jain, and Inderjit S. Dhillon. "Efficient matrix sensing using rank-1 gaussian measurements", 2015, Banff, AB, Canada, October 2015, ALT, 3-18.
21. Lu, Weizhi, Weiyu Li, Kidiyo Kpalma, and Joseph Ronsin."Compressed sensing performance of random Bernoulli matrices with high compression ratio", 2015, IEEE Signal Processing Letters, vol. 22, No.8, 1074-1078.

22. Tiwari, Vibha, P. P. Bansod, and Abhay Kumar. "Designing sparse sensing matrix for compressive sensing to reconstruct high resolution medical images", 2015, Cogent Engineering vol. 2, No.1, 1017244-1017255.