Advances in Modelling and Analysis A
Vol. 55, No. 2, June, 2018, pp. 82-87
Journal homepage: http://iieta.org/Journals/AMA/AMA_A

The Cole-Hopf transformation for solving Rayleigh-Plesset equation in bubble dynamics

Ali F. Abu-Bakr[1,2*], Aly M. Abourabia[1]

[1] Mathematics and Computer Science Department, Menoufia University, Shebin El-Kom 32511, Egypt
[2] Theoretical and Mathematical Physics Department, Ural Federal University, Ekaterinburg 620083, Russia

Corresponding Author Email: alibakrm@yahoo.com

Received: March 20 2018
Accepted: May 18 2018

Keywords:
Cole–Hopf transformation, Rayleigh-Plesset equation, phase portrait, bubble dynamics

ABSTRACT

In this paper, we investigate the solution of the behaviour of bubble growth in Newtonian fluid by using extended Rayleigh-Plesset equation. The transformation of Cole-Hopf has been applied on the nonlinear ordinary differential equation of the non-dimensional extended Rayleigh-Plesset equation in order to obtain the exact solution of bubble radius. We have also introduced the study of phase portrait of growth problem. The results studied analytically and indicated in graphics.

1. INTRODUCTION

It is well known that the study of nonlinear differential equations of Rayleigh-Plesset equation plays an important role in various areas of applied mathematics and theoretical physics like biomedical treatments [1], sonoluminescing bubbles [2], bubbles in turbulent flow [3-4], vortical flow field [5], cavitating water jets [6], The exact determination of the thermal term in the Rayleigh–Plesset equation [7-8] needs to solve the diffusion equation that leads to significant difficulties due to its nonlinearities. For reduction in the complexity of the bubble growth problem, Plesset and Zwick [9] considered two limiting regions of bubble growth. Plesset and Zwick [10-11] independently determined that the bubble growth is thermally controlled by the rate of energy which is transferred through the liquid to the vapour liquid interface.

Many authors are interested in studying nonlinear processes in a liquid with dynamics of vapour or gas bubbles by the Plesset-Rayleigh numerically (see, e.g. [6, 11-12]) and other analytically (see, e.g. [3, 13-15]). For an example, author in [13] studied dynamics of the Rayleigh-Plesset equation modeling a gas-filled bubble immersed in an incompressible fluid. However, analytical solutions can be useful for the investigation of bubbles dynamics and developing its applications.

The aim of this paper is to study the exact solution of Rayleigh-Plesset equation in some domain with initial values by the method of Cole–Hopf transformation which transforms from a non-linear ordinary differential equation into linearly differential equation. We then formally derive the exact solutions of Rayleigh-Plesset equation. This goal use to develop a theoretical model with an analytical solution that describes the physics and behaviour of growth bubble.

The structure of the paper is the following. Section 2 introduces the mathematical formulation of non-dimensional extended Rayleigh-Plesset equation. Section 3 presents the Cole Hopf transformation and the analytical solution of non-dimensional extended Rayleigh-Plesset equation. The phase portrait of non-dimensional extended Rayleigh-Plesset equation is presented in section 4. Section.5 is devoted to the discussion of results. Finally, the conclusions are introduced in section 6.

2. MATHEMATICAL MODEL

A single vapour bubble is considered to grow inside a mixture of vapour and superheated liquid between two finite radius boundaries R_0 and R_m. The problem is affected by some parameters such as the pressure $\overline{\Delta P}$ between the bubble pressure P_v and P_∞, and temperature difference between the phases and other physical parameters. The problem is shown in Fig. 1.

Moreover, we consider that the bubble is assumed to have a spherical geometry. The mixture of vapour and superheated liquid are assumed as an incompressible. Pressure inside the bubble is assumed to be uniform. Vapour density distribution inside the bubble is assumed to be uniform except for a thin boundary layer near the bubble wall. The viscosity of the fluid is considered. The equation of motion of bubble is without gravitational effects.

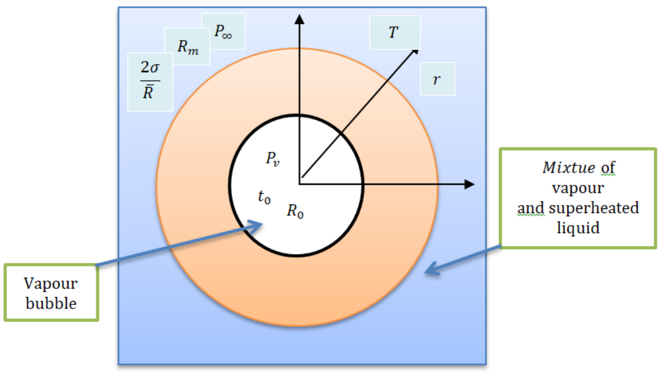

Figure 1. Sketch of bubble dynamic.

On a basis of Navier-Stoke's equation, the extended Rayleigh-Plesset equation describes the bubble dynamics problem. Then, the equation of motion for a spherically symmetric bubble takes the form:

$$\bar{R}\ddot{\bar{R}}+b\dot{\bar{R}}^2=\frac{1}{\rho}\left[\Delta\bar{P}-\frac{2\sigma}{\bar{R}}\right]-\frac{4\eta\,\dot{\bar{R}}}{\rho\,\bar{R}}. \tag{1}$$

Here, $b=\frac{3}{2}$, ρ is a density of a liquid, σ is a surface tension, η is a viscosity, $\bar{R}$ is the bubble radius as a function of time and dot denotes time differentiating and $\dot{\bar{R}}=\frac{d\bar{R}}{d\bar{t}}$ is instantaneous radial velocity of bubble boundary.

We can be introduced the extended Rayleigh-Plesset equation (1) in non-dimensional form, so we suppose that R^*, t^*, P^* are characteristic scales of radius $\bar{R}$, time $\bar{t}$, and pressure P, respectively, and defined as

$$r=\frac{\bar{R}}{R^*} \qquad t=\frac{\bar{t}}{t^*}$$
$$P=\frac{\bar{P}}{P^*} \tag{2}$$

Substituting from relations (2) into equation (1), then

$$r\ddot{r}+b\dot{r}^2=\left[\frac{P^*t^{*2}}{\rho\,R^{*2}}\frac{\Delta P}{P^*}-\frac{2\sigma\,t^{*2}}{\rho\,R^{*3}r}\right]-\frac{4\eta\,t^*\dot{r}}{\rho\,R^{*2}r}. \tag{3}$$

The above equation (3) takes the non-dimensional form

$$r\ddot{r}+b\dot{r}^2=\left[Th\frac{\Delta P}{P^*}-\frac{We}{r}\right]-\frac{1}{\mathrm{Re}}\frac{\dot{r}}{r}, \tag{4}$$

with the following set of non-dimensional numbers:

- Thoma cavitation number, $Th=\dfrac{P^*t^{*2}}{\rho\,R^{*2}}$.
- Weber number, $We=\dfrac{2\sigma\,t^{*2}}{\rho\,R^{*3}}$,
- Reynolds number, $\mathrm{Re}=\dfrac{R^{*2}}{4\nu\,t^*}$,

where, the kinematical viscosity has the form: $\nu=\frac{\eta}{\rho}$.

The scaling is well suited to the problem if the values of r and t are of the order of unity. For non-irregular behaviour of the physical phenomenon, the quantity $\dot{r}$ an $\ddot{r}$ are consequently also of the order of unity. Then, the above non-dimensional numbers allow us to compare the importance of different terms in equation (4): pressure, surface tension and viscosity.

The choice of the reference length and time scales deserves caution as the radius and the velocities can change by several orders of magnitude during the bubble evolution. Thus, in general, the scaling is well adapted to a limited phase of the phenomenon only.

We can rewrite the equation (4) in the form

$$r^2\frac{d^2r}{dt^2}+b\,r\left(\frac{dr}{dt}\right)^2+a\frac{dr}{dt}-c\,r+d=0, \tag{5}$$

where,

$a=\frac{1}{Re}$, $b=\frac{3}{2}$, $c=\frac{\Delta P\,Th}{P^*}$ and $d=We$.

3. THE COLE-HOPF TRANSFORMATION AND THE ANALYTICAL SOLUTION OF RAYLEIGH-PLESSET EQUATION

On the basis of Cole-Hopf transformation that was used in studies of nonlinear ordinary differential equation and in mechanics (see, refs. [16-18], in this section, we will demonstrate the Cole–Hopf transformation method that used to solve non-dimensional extended Rayleigh-Plesset equation (5).

To simplify equation (5), we suppose that the displacement r, is defined in $r=R+\frac{d}{c}$, then we get

$$\left(\frac{d}{c}+R\right)^2\frac{d^2R}{dt^2}+b\left(\frac{d}{c}+R\right)\left(\frac{dR}{dt}\right)^2+a\frac{dR}{dt}-c\,R=0. \tag{6}$$

Introducing the Cole–Hopf transformation in the form: $R=\alpha\frac{d^2}{dt^2}\mathrm{Log}(\phi(t))+\beta\frac{d}{dt}\mathrm{Log}(\phi(t))$, that gives us in another form

$$R=\alpha\left(-\frac{\phi'(t)^2}{\phi(t)^2}+\frac{\phi''(t)}{\phi(t)}\right)+\beta\frac{\phi'(t)}{\phi(t)} \tag{7}$$

where, α and β are determined later.

The first and the second derivatives would have the form:

$$\frac{dR}{dt}=\frac{1}{\phi(t)^3}(2\alpha\phi'(t)^3-\phi(t)\phi'(t)(\beta\phi'(t)+3\alpha\phi''(t))+\phi(t)^2(\beta\phi''(t)+\alpha\phi^{(3)}(t))), \tag{8}$$

and

$$\frac{d^2R}{dt^2}=\frac{1}{\phi(t)^4}(-6\alpha\phi'(t)^4+2\phi(t)\phi'(t)^2(\beta\phi'(t)+6\alpha\phi''(t))-\phi(t)^2(3\alpha\phi''(t)^2+\phi'(t)(3\beta\phi''(t)+4\alpha\phi^{(3)}(t)))+\phi(t)^3(\beta\phi^{(3)}(t)+\alpha\phi^{(4)}(t))). \tag{9}$$

Hence, using the transformation (7), and its derivatives (8) and (9) in (6). The equation (6) is reduced to the following:

$$-c\left(\frac{\beta\phi'(t)}{\phi(t)}+\alpha\left(-\frac{\phi'(t)^2}{\phi(t)^2}+\frac{\phi''(t)}{\phi(t)}\right)\right)+a\left(-\frac{\beta\phi'(t)^2}{\phi(t)^2}+\frac{\beta\phi''(t)}{\phi(t)}+\alpha\left(\frac{2\phi'(t)^3}{\phi(t)^3}-\frac{3\phi'(t)\phi''(t)}{\phi(t)^2}+\frac{\phi^{(3)}(t)}{\phi(t)}\right)\right)+b\left(\frac{d}{c}+\frac{\beta\phi'(t)}{\phi(t)}+\alpha\left(-\frac{\phi'(t)^2}{\phi(t)^2}+\frac{\phi''(t)}{\phi(t)}\right)\right)\left(-\frac{\beta\phi'(t)^2}{\phi(t)^2}+\frac{\beta\phi''(t)}{\phi(t)}+\alpha\left(\frac{2\phi'(t)^3}{\phi(t)^3}-\frac{3\phi'(t)\phi''(t)}{\phi(t)^2}+\frac{\phi^{(3)}(t)}{\phi(t)}\right)\right)^2+\left(\frac{d}{c}+\frac{\beta\phi'(t)}{\phi(t)}+\alpha\left(-\frac{\phi'(t)^2}{\phi(t)^2}+\frac{\phi''(t)}{\phi(t)}\right)\right)^2\left(\frac{2\beta\phi'(t)^3}{\phi(t)^3}-\frac{3\beta\phi'(t)\phi''(t)}{\phi(t)^2}+\frac{\beta\phi^{(3)}(t)}{\phi(t)}+\alpha\left(-\frac{6\phi'(t)^4}{\phi(t)^4}+\frac{12\phi'(t)^2\phi''(t)}{\phi(t)^3}-\frac{3\phi''(t)^2}{\phi(t)^2}-\frac{4\phi'(t)\phi^{(3)}(t)}{\phi(t)^2}+\frac{\phi^{(4)}(t)}{\phi(t)}\right)\right)=0. \tag{10}$$

From software program (like as Mathematica software (version 11.0)), we can calculate the power classification for ϕ^i, $-1\le i\le -8$ that is to be

$\phi(t)^{-1}$:

$$-c\beta\phi'(t)-c\alpha\phi''(t)+a\beta\phi''(t)+a\alpha\phi^{(3)}(t)+\frac{d^2\beta\phi^{(3)}(t)}{c^2}+\frac{d^2\alpha\phi^{(4)}(t)}{c^2}=0, \tag{11a}$$

$\phi(t)^{-2}$:

$$\frac{1}{c^2}(c^2(c\alpha-a\beta)\phi'(t)^2+d((-3d\alpha+bc\beta^2)\phi''(t)^2+bc\alpha^2\phi^{(3)}(t)^2+2c\alpha\phi''(t)((1+b)\beta\phi^{(3)}(t)+\alpha\phi^{(4)}(t)))+\phi'(t)(-3(ac^2\alpha+d^2\beta)\phi''(t)+2d((-2d\alpha+c\beta^2)\phi^{(3)}(t)+c\alpha\beta\phi^{(4)}(t))))=0, \tag{11b}$$

$\phi(t)^{-3}$:

$$\frac{1}{c^2}(2(ac^2\alpha+d^2\beta)\phi'(t)^3+c\alpha\phi''(t)((-6d\alpha+bc\beta^2)\phi''(t)^2+bc\alpha^2\phi^{(3)}(t)^2+c\alpha\phi''(t)((\beta+2b\beta)\phi^{(3)}(t)+\alpha\phi^{(4)}(t)))+c\phi'(t)((-6(2+b)d\alpha\beta+bc\beta^3)\phi''(t)^2+bc\alpha^2\beta\phi^{(3)}(t)^2+2\alpha\phi''(t)((-(4+3b)d\alpha+(1+b)c\beta^2)\phi^{(3)}(t)+c\alpha\beta\phi^{(4)}(t)))+\phi'(t)^2(2d(6d\alpha-(3+b)c\beta^2)\phi''(t)+c((-2(5+b)d\alpha\beta+c\beta^3)\phi^{(3)}(t)+\alpha(-2d\alpha+c\beta^2)\phi^{(4)}(t)))))=0, \tag{11c}$$

$\phi(t)^{-4}$:

$$\frac{1}{c^2}(d(-6d\alpha+(4+b)c\beta^2)\phi'(t)^4-3c^2\alpha^3\phi''(t)^4-c^2\alpha^2\phi'(t)\phi''(t)^2(3(3+2b)\beta\phi''(t)+2(2+3b)\alpha\phi^{(3)}(t))-c\alpha\phi'(t)^2((-3(10+3b)d\alpha+9(1+b)c\beta^2)\phi''(t)^2+bc\alpha^2\phi^{(3)}(t)^2+2c\alpha\phi''(t)(5(1+b)\beta\phi^{(3)}(t)+\alpha\phi^{(4)}(t)))+c\phi'(t)^3(\beta(2(17+5b)d\alpha-(3+2b)c\beta^2)\phi''(t)+2\alpha((2(2+b)d\alpha-(3+b)c\beta^2)\phi^{(3)}(t)-c\alpha\beta\phi^{(4)}(t))))=0, \tag{11d}$$

$\phi(t)^{-5}$:

$$\frac{1}{c}\phi'(t)^2((-4(4+b)d\alpha\beta+(2+b)c\beta^3)\phi'(t)^3+9(2+b)c\alpha^3\phi''(t)^3+c\alpha^2\phi'(t)\phi''(t)((38+25b)\beta\phi''(t)+2(4+5b)\alpha\phi^{(3)}(t))+\alpha\phi'(t)^2((-12(3+b)d\alpha+(22+13b)c\beta^2)\phi''(t)+c\alpha(3(3+2b)\beta\phi^{(3)}(t)+\alpha\phi^{(4)}(t))))=0, \tag{11e}$$

$\phi(t)^{-6}$:

$$-\frac{1}{c}\alpha\phi'(t)^4((-4(3+b)d\alpha+5(2+b)c\beta^2)\phi'(t)^2+3(11+7b)c\alpha^2\phi''(t)^2+c\alpha\phi'(t)((43+26b)\beta\phi''(t)+4(1+b)\alpha\phi^{(3)}(t)))=0, \tag{11f}$$

$\phi(t)^{-7}$:

$$2\alpha^2\phi'(t)^6((7+4b)\beta\phi'(t)+4(3+2b)\alpha\phi''(t))=0, \tag{11g}$$

$\phi(t)^{-8}$:

$$-2(3+2b)\alpha^3\phi'(t)^8=0. \tag{11h}$$

By solving equation (11g), the solution of differential equation is

$$\phi(t)=-\frac{4(3+2b)e^{-\frac{(7+4b)t\beta}{4(3+2b)\alpha}}\alpha C_1}{(7+4b)\beta}+C_2. \tag{12}$$

Substituting from (112) into (7), we get

$$R(t)=-\frac{(7+4b)K\beta^2(16(3+2b)^2K\alpha-(5+4b)(7+4b)e^{\frac{7t\beta+4bt\beta}{12\alpha+8b\alpha}}\beta)}{4(3+2b)(-4(3+2b)K\alpha+(7+4b)e^{\frac{7t\beta+4bt\beta}{12\alpha+8b\alpha}}\beta)^2}. \tag{13}$$

Here, $K=\frac{C_1}{C_2}$; is a constant, we can determine K under these initial conditions: $t=0$ and $R(0)=R_0$, We obtain K in the form

$$K=\frac{-aK_1c^4d+(569c+1077bc+680b^2c+144b^3c)\sqrt{K_3}-4\sqrt{K_4}}{K_5((a^2c+d^2)K_2c^2d-a\sqrt{K_3})}, \tag{14}$$

where K_1, K_2, K_3 and K_4 are defined as

$$K_1=12869073+70355331b+170650961b^2+240926173b^3+218063862b^4+131134456b^5+52352192b^6+13366272b^7+1977856b^8+129024b^9,$$

$$K_2=22617+80838b+119875b^2+94190b^3+41240b^4+9504b^5+896b^6,$$

$$K_3=(K_2)^2c^5d^2(a^2c+4d^2),$$

$$K_4=(-(3+b)^2(7+4b)^3(152934+717349b+1452609b^2+1645874b^3 1126752b^4+465904b^5+107776b^6+10752b^7)c^5d((a^2c+d^2)K_2c^2d-a\sqrt{K_3}),$$

$$K_5=1367+2303b+1288b^2+240b^3.$$

To find α and β, we use power classification $\phi(t)^{-6}$ in (11f), α takes

$$\alpha=-\frac{(359+719b+488b^2+112b^3)c\beta^2}{64(3+b)(3+2b)^2d}, \tag{15}$$

and

$$256(7+4b)^2d^4+\frac{16a(7+4b)\big(359+b\big(719+8b(61+14b)\big)\big)c^3d\beta}{(3+b)(3+2b)}-\frac{\big(359+b\big(719+8b(61+14b)\big)\big)^2c^5\beta^2}{(9+9b+2b^2)^2}=0. \tag{16}$$

We get the solution of the equation (16) in the form

$$\beta=\frac{\beta\beta_1+\sqrt{(\beta\beta_1)^2+4(\beta\beta_2)(\beta\beta_3)}}{2\,(\beta\beta_2\,)}, \tag{17a}$$

Or

$$\beta=\frac{\beta\beta_1-\sqrt{(\beta\beta_1)^2+4(\beta\beta_2)(\beta\beta_3)}}{2\,(\beta\beta_2\,)}, \tag{17b}$$

where,

$$\beta\beta_1=361872ac^3d+1293408abc^3d+1918000ab^2c^3d+1507040ab^3c^3d+659840ab^4c^3d+152064ab^5c^3d+14336ab^6c^3d,$$

$$\beta\beta_2=128881c^5+516242bc^5+867345b^2c^5+782160b^3c^5+109312b^5c^5+12544b^6c^5,$$

and

$$\beta\beta_3=1016064d^4+3193344bd^4+4121856b^2d^4+2792448b^3d^4+1045504b^4d^4+204800b^5d^4+16384b^6d^4.$$

The solution of displacement of bubble growth, r takes the form

$$r(t) = \frac{8(7+4b)a\psi_1 - \psi_2\left(\psi_3 e^{\frac{-2\psi_1 ct}{a\psi_1-\psi_2}} + \frac{2a(3+b)}{d}\psi_1 K - \frac{2}{(3+b)d}\psi_2\right)}{\psi_4\left(2\psi_5 e^{\frac{-2\psi_1 ct}{a\psi_1-\psi_2}} + \frac{a}{c}\psi_5 K - \frac{1}{(3+b)(3+2b)d}\psi_2 K\right)^2} + \frac{d}{c}, \tag{18}$$

where,

$\psi_1 = (3+b)(7+4b)(359+b(719+8b(61+14b)))c^3 d$,

$\psi_2 = \sqrt{\frac{a^2c+4d^2}{c}}K$,

$\psi_3 = (5+4b)(7+4b)(359+b(719+8b(61+14)))$,

$\psi_4 = \frac{c(3+2b)}{(5+4b)(7+4b)}\psi_3$,

$\psi_2 = \frac{c^4}{(5+4b)}\psi_5$.

4. THE PHASE PORTRAIT OF EXTENDED RAYLEIGH-PLESSET EQUATION

In the follows section 2, the non-dimensional extended Rayligh-Plesset equation (5) was introduced in non-dimensional. And equation (5) can be solved by using phase plane, so we suppose that

$$x = r \text{ and } y = \dot{r}. \tag{19}$$

The derivatives of above relations take the form

$$\begin{aligned} \dot{x} &= y = \dot{r} = f_1(x,y). \\ \ddot{x} &= \dot{y} = \ddot{r} = f_2(x,y). \end{aligned} \tag{20}$$

From equations (19, 20) into equation (5), then equation (5) leads to

$$x^2\dot{y} = -b\,x\,y^2 - a\,y + c\,x - d. \tag{21}$$

The $\dot{y}$ becomes in the form

$$\dot{y} = -\frac{b\,y^2}{x} - \frac{a\,y}{x^2} + \frac{c}{x} - \frac{d}{x^2} = f_2(x,y). \tag{22}$$

To obtain the critical points, we put $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} = 0$, then

$$\frac{d}{x} - \frac{c}{x^2} = 0 \;\Rightarrow x = \frac{d}{c}. \tag{23}$$

The equilibrium point is $(\frac{d}{c}, 0)$.

To estimate the eigenvalues, we proceed as follows; the Jacobian matrix

$$J = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{b\,x\,y^2 - cx + 2a\,y + 2d}{x^3} & -\frac{2b\,x\,y + a}{x^2} \end{vmatrix}. \tag{24}$$

The eigenvalues have the form

$$\lambda_1 = \frac{-a\,x - 2b\,x^2 y - \sqrt{(a\,x + 2b\,x^2 y)^2 - 4x^3(-2d + c\,x - 2ay - b\,x\,y^2}}{2x^3}, \tag{25}$$

and

$$\lambda_2 = \frac{-a\,x - 2b\,x^2 y + \sqrt{(a\,x + 2b\,x^2 y)^2 - 4x^3(-2d + c\,x - 2ay - b\,x\,y^2}}{2x^3}. \tag{26}$$

By using values of physical parameters in the situation, the eigenvalues can be found at critical point $(1070, 0)$ in the form:

$\lambda_1 = -0.000954715$ and $\lambda_2 = 0.000954679$.

Which give us an unstable saddle at $(1070, 0)$ and $\lambda_1\lambda_2 < 0$, With the help of the Mathematica software, the phase portrait classifications of the solutions of the autonomous system (20) and (22) are illustrated parametrically as a curve in the xy-plane, see Fig. 5.

5. RESULTS AND DISCUSSION

The extended of Rayleigh-Plesset equation (1) is transformed to non-dimensional equation in (5) that equation (5) is solved analytically using Cole-Hopf transformation in (7). The analytical solution of non-dimensional extended of Rayleigh-Plesset equation is given in equation (18). The presented solution in (18) represents the displacement of bubble growth and the effect of the physical parameters on the growth process. The solution of equation (5) is depended on convenient value of β, so we choose a convenient value of β in equation (17a) that give us a physical sense of the behaviour of bubble dynamics.

The numerical values of physical parameters, which are used in the simulating and calculations for the bubble growth as:

$\eta = 10^{-5} kgm^{-1}$, $P^* = 247 kgm^{-1}s^{-2}$, $\overline{\overline{\Delta P}} = 10 kgm^{-1}s^{-2}$, $R^* = 10^{-5} m$, $t^* = 10^{-4} s$, $We = 1116.5$, $Th = 128.87$, $Re = 23.95$.

Moreover, Fig.2 shows the comparison between the solution of equation (18) and $r = \frac{d}{c}$. We note that the maximum of the growth process at the surface of bubble, that means $r = \frac{d}{c}$.

The effect of physical parameters on the bubble growth is calculated by Mathematica program (Version (11.0)). In Fig. 2, the displacement of bubble growth as a function of time is given by solving the non-dimensional of extended of Rayleigh–Plesset equation (5) with a Cole Hopf transformation in (7). We note that the result gives us the growth of bubble.

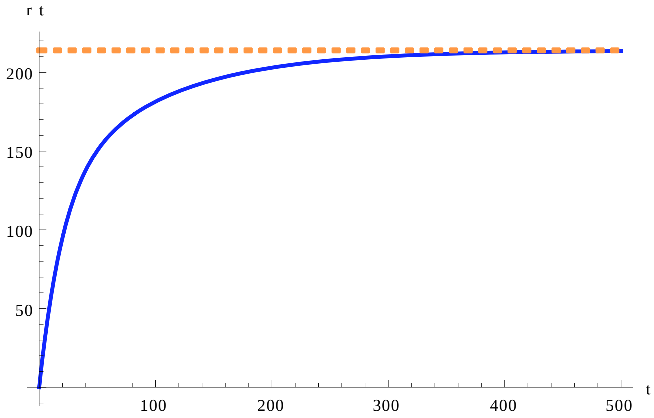

Figure 2. The growth of the bubble radius as a function of time. Blue curve represents of the solution of equation (18) and orange-dashed line represents $r = \frac{d}{c}$

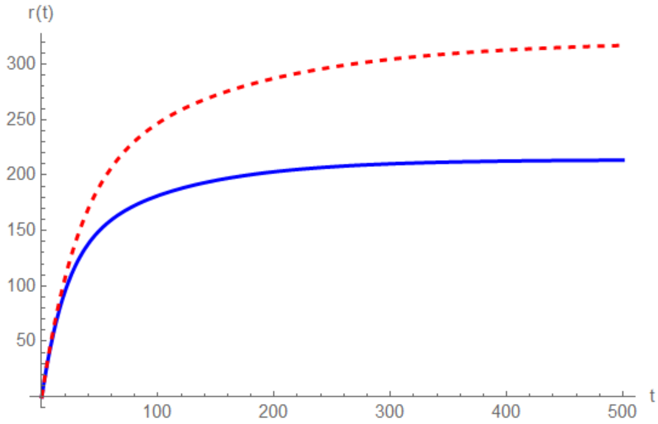

Figure 3. The displacement of bubble growth as a function of time under effect Weber Number, We. Blue curve - We, dashed red curve - 1.5 We

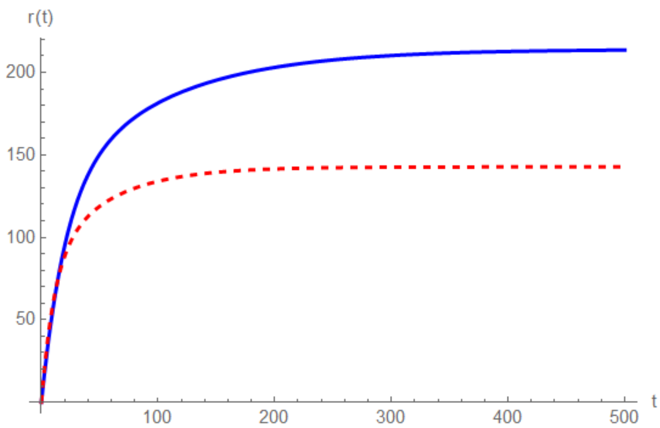

Figure 4. The displacement of bubble growth as a function of time under effect Thoma number, Th. Blue curve – Th, dashed red curve: 1.5 Th

Fig. 3 explains the effect of increasing the value of the Weber number We on the bubble radius throughout the growth period, while it, the growth bubble is inversely proportional with Thoma cavitation number Th as shown in Fig. 4. The phase portrait of the solutions of dimensionless of Rayleigh-Plesset equation is illustrated in Fig. 5, gives us an unstable saddle.

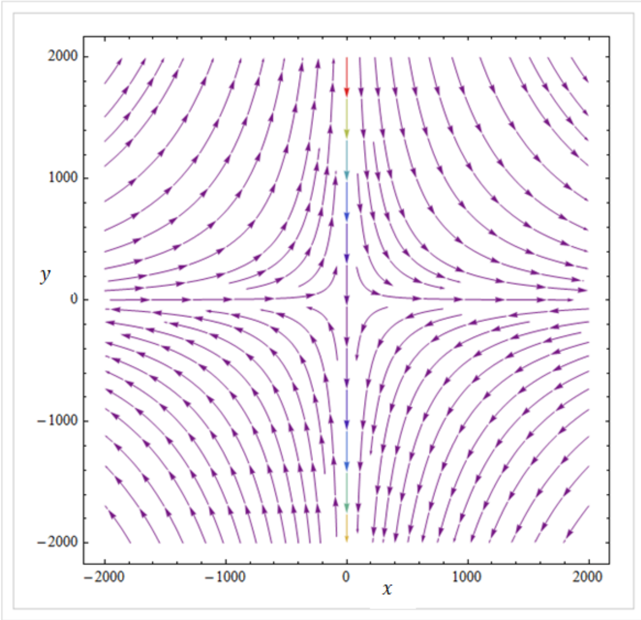

Figure 5. The phase plane of non-dimensional extended Rayleigh-Plesset equation (5)

6. CONCLUSIONS

We employed the Cole-Hopf transformation for finding the analytical solution of extended of Rayleigh-Plesset equation. The method has been shown to computationally efficient in solving the equation of growth bubble. The growth of bubble radius is proportional to Weber number and Thoma number.

The problem is studied in phase plane in non-dimensional equation and the system of the bubble growth is unstable saddle point. These results must be taken into account while developing some applications of the bubble dynamics such as biomedical applications.

REFERENCES

[1] Yang X, Church C. (2005). A model for the dynamics of gas bubbles in soft tissue. The Journal of the Acoustical Society of America 118: 3595-3606. http://dx.doi.org/10.1121/1.2118307

[2] Hilgeneedt S, Brenner M, Grossmann S, Lohse D. (1998). Analysis of Rayleigh-Plesset dynamics for sonoluminescing bubbles. Journal of Fluid Mechanics 365: 171-204. https://doi.org/10.1017/S0022112098001207

[3] Mohammadein SA. Abu-Bakr AF. (2010). The growth of vapour bubble in a superheated liquid between two-phase turbulent flow. Canadian Journal of Physics 88(5): 317-324. http://dx.doi.org/10.1139/P10-022

[4] Guzman DN, Friczek T, Reetz C, Sun C, Lohse D, Ahlers G. (2016). Vapour-bubble nucleation and dynamics in turbulent Rayleigh_Bénard convection. Journal of Fluid Mechanics 795: 60-95. https://doi.org/10.1017/jfm.2016.178

[5] Cui B, Ni B, Wu Q. (2016). Bubble–bubble interaction effects on dynamics of multiple bubbles in a vortical flow field. Advances in Mechanical Engineering 8(2): 1–12. https://doi.org/10.1177/1687814016631708

[6] Alehossein H, Qin Z. (2007). Numerical analysis of Rayleigh–Plesset equation for cavitating water jets. International Journal Numerical Mechanics Engineering 72: 780–807. https://doi.org/10.1002/nme.2032

[7] Rayleigh L. (1917). On the pressure developed in a liquid during the collapse of a spherical cavity. Philosophical Magazine Series 6, 34: 94–98. http://dx.doi.org/10.1080/14786440808635681

[8] Plesset MS. (1949). The dynamics of cavitating bubbles. Journal of Applied Mechanics 16: 228–231. http://resolver.caltech.edu/CaltechAUTHORS:20140808-114249321

[9] Plesset MS, Zwick SA. (1954). The growth of vapor bubbles in superheated liquids. Journal of Applied Mechanics 25(4): 493–450. http://dx.doi.org/10.1063/1.1721668

[10] Prosperetti A. (1982). A generalization of the Rayleigh-Plesset equarion of bubble dynamics. Physics Fluids 25(3): 409-410. https://doi.org/10.1063/1.863775

[11] Klaseboer E, Khoo BC. (2006). A modified Rayleigh–Plesset model for a non-spherically symmetric oscillating bubble with applications to boundary integral methods. Engineering Analysis with Boundary Elements 30: 59–71. https://doi.org/10.1016/j.enganabound.2005.09.003

[12] Johnsen E, Colonus T. (2009). Numerical simulations of non-spherical bubble collapse. Journal of Fluid Mechanics 629: 231–262. https://doi.org/10.1017/S0022112009006351

[13] Van-Gorder RA. (2016). Dynamics of the Rayleigh-Plesset equation modelling a gas-filled bubble immersed in an incompressible fluid, Journal of Fluid Mechanics 807: 478-508. https://doi.org/10.1017/jfm.2016.640

[14] Mohammadein SA, Shalaby GA, Abu-Bakr AF, Abu-Nab AK. (2017). Analytical solution of gas bubble dynamics between two-phase flow. Results in Physics 7: 2396-2403. http://dx.doi.org/10.1016/j.rinp.2017.07.007

[15] Kudryashov NA, Sinelshchikov DI. (2015). Analytical solutions for problems of bubble dynamics. Physics Letters A 379 (8): 798–802. http://dx.doi.org/10.1016/j.physleta.2014.12.049

[16] Abourabia AM, El Horbaty MM. (2006). On solitary wave solutions for the two-dimensional nonlinear modified Kortweg–de Vries–Burger equation. Chaos, Solitons and Fractals 29: 354–364. https://doi.org/10.1016/j.chaos.2005.08.112

[17] Gorguis A. (2006). A comparison between Cole–Hopf transformation and the decomposition method for solving Burgers' equations. Applied Mathematics and Computation, 173: 126–136. https://doi.org/10.1016/j.amc.2005.02.045

[18] Wazwaz A. (2012). A study on the (2+1)-dimensional and the (2+1)-dimensional higher-order Burgers equations. Applied Mathematics Letters 25: 1495–1499. https://doi.org/10.1016/j.aml.2011.12.034