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A Robust Block Hybrid Trigonometric Method for the Numerical Integration of Oscillatory Second Order Nonlinear Initial Value Problems.

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Abstract

A new Block Hybrid Trigonometrically Fitted Method (BHTM) for the numerical integration of second order nonlinear initial value problems with oscillatory solutions is presented in this paper. The BHTM is based on multistep collocation method. The examination of the stability properties of the method shows that it is A-stable. Numerical experiments are carried out to show the accuracy and efficiency of the method on second order nonlinear initial value problems with oscillatory solutions.

Keywords

Collocation, Hybrid, Nonlinear Second Order IVPs, Trigonometrically Fitted.

1. Introduction

In the past two decades there has been considerable interest in effective numerical integration of the initial value problem of second-order differential equations in the form

$$\left.\begin{aligned} y''(x) = f(x, y(x)) \ , \qquad x \in [x_0\,, b] \\ y(x_0) = y_0, \ \ y'(x_0) = y_0' \end{aligned}\right\} \qquad (1)$$

which solution has oscillatory characteristics, where $y \in \mathbb{R}^n$,$f\colon [x_0\,, b] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is sufficiently differentiable, satisfies Lipschitz condition and the first derivative y' does not appear explicitly. Such problems arise in area such as quantum mechanics, celestial mechanics, and theoretical chemistry among others. Most of these problems are nonlinear, and as a result they are often extremely difficult, or sometimes impossible, to solve analytically with presently available mathematical methods. With respect to the oscillatory feature of the problem (1), researchers have proposed integrators with frequency-dependent coefficients that exactly integrate a set of linearly independent non polynomial function for the solution of (1). Some of the methods include trigonometric polynomial interpolation (Gautschi, 1961), mixed interpolation (De Meyer *et al.*, 1990; Coleman and Duxbury, 2000; Vanthornout *et al.*, 1990), exponential fitting methods (Ixaru *et al.*, 2002; Vaden Berghe *et al.*, 1999; Van Berghe and Van Daele, 2007; Simos 1998 and 2002) functional fitting (Ozawa, 2001), Piecewise Linearized methods (Ramos, 2006). More recently, in the context of continuous multistep collocation method for the construction of trigonometrically fitted methods, Ngwane and Jator (2013a&b and 2015a) proposed block hybrid method, Ngwane and Jator (2015b) considered Enright method and Ndukum *et al.*, (2017) proposed extended backward differentiation.

This paper considers a basis function other than polynomial for the development of a hybrid method via multistep collocation method. According to Duxbury (1999), one incentive for using a basis function other than polynomial is the fact that as every oscillation has to be followed when integrating oscillatory IVP, then a large amount of computer time is required and the rounding error accumulates for small sizes. Methods based on polynomial functions are therefore not reliable. In view of this, basis function in this research is the set $\{1, x, \dots, x^n, \cos(\omega x), \sin(\omega x)\}$. This is motivated because of its simplicity to analyses (Ngwane and Jator, 2015b) and better approximation for initial value problems with oscillatory solution (Coleman and Duxbury, 2000).

The collocation methods for ordinary differential equation are based on a simple algorithm, find a function of a specified form that satisfies the differential equation exactly at a given set of points. Basically, collocation method is the bedrock of continuous schemes. Some of the advantages of continuous form of linear multistep method over discrete method include better error estimation, provision of a simplified coefficient for further analytical work at different points, provision of approximations at all interior points (Awoyemi, 1999) and ability to generate infinite number of schemes (Oluwatosin, 2013).

The idea of block methods started with Milne (1953), who use this idea only as a mean of generating starting values for predictor-corrector algorithms. Rosser (1967), developed the block LMM into methods for solving IVPs. Block methods contain two parts viz: main and complementary methods (Brugnano and Trigiante, 1998). Some of the advantages of block methods include but not limited to permission of easy change of step length (Yakusak and

Adeniyi, 2015), self-starting and thus avoiding the use of other method to get the starting solution, overcoming the overlapping of pieces of solution and obtaining numerical solution at more than one point at a time (Ramos and Singh, 2017).

Usually, the hybrid method is compounded with the need to develop predictors for the evaluation of the corrector at off grid points making the method time consuming and more tedious (Akinfenwa *et al.*, 2011). It is our aim to show that the block trigonometrically fitted hybrid method in this paper can be made to overcome this shortcoming and cope with the integration of nonlinear oscillatory problems.

2. Derivation of the method

The proposed Block Hybrid Trigonometric Method (BHTM) with symmetric hybrid points is of the form

$$\sum_{j=0}^{1} \alpha_j(u) y_{n+j} + \sum_{m=1}^{2} \alpha_{vm}(u)\, y_{n+vm} = h\beta_2(u)\, f_{n+2} + h^2\delta_2(u) g_{n+2} \qquad (2)$$

where $y_{n+j} = y(x_n + jh)$, $f_{n+2} = y'(x_n + 2h)$, $g_{n+2} = y''(x_n + 2h)$, $u = \omega h$, ω is the frequency, x_n is a node point, $vm = \left\{\frac{1}{2}, \frac{3}{2}\right\}$ is the hybrid point and $\alpha_j, \beta_j, \delta_k$ are parameter to be uniquely obtained from multistep collocation technique and dependent on the step size and frequency.

The coefficients of BHTM are selected so that the method integrates the IVP (1) exactly where the solutions are members of the linear space $\{1, x, x^2, x^3, \sin(\omega x), \cos(\omega x)\}$.

Through interpolation of the basis function given by

$$y(x,u) = \sum_{j=0}^{3} a_j x^j + a_4 \sin(\omega x) + a_5 \cos(\omega x)\,, \quad \omega = \frac{u}{h}$$

at the points x_{n+j}, and x_{n+vm} $j = 0{,}1$ and $m = 1{,}2$ respectively, collocation $\frac{\partial(y(x,u))}{\partial x}$ at the point x_{n+2} and collocation of $\frac{\partial^2(y(x,u))}{\partial x^2}$ at the points x_{n+2}, the Continuous Block Hybrid Trigonometric Method (CBHTM) which will be used to produce the main discrete formula and its derivative will be used to generate three additional discrete formulas for solving equation (1) is obtained as

$$y(x,u)=\sum_{j=0}^{1}\alpha_j(x,u)y_{n+j}+\sum_{m=1}^{2}\alpha_{vm}(x,u)\,y_{n+vm}+h\beta_2(x,u)f_{n+2}+h^2\delta_k(x,u)g_{n+2} \quad (3)$$

where $\alpha_0(x,u),\alpha_{v1}(x,u),\alpha_1(x,u),\alpha_{v2}(x,u),\beta_2(x,u)$ and $\gamma_2(x,u)$ are continuous coefficients that satisfy the root condition.

Theorem 1

Let equation (3) satisfies the following equations:

$$y(x_{n+j},u) = y_{n+j}\ ,\quad j=0\left(\tfrac{1}{2}\right)\tfrac{3}{2} \quad (4)$$

$$\left.\frac{\partial(y(x,u))}{\partial x}\right|_{x=x_{n+2}} = f_{n+2} \quad (5)$$

$$\left.\frac{\partial^2(y(x,u))}{\partial x^2}\right|_{x=x_{n+2}} = g_{n+2} \quad (6)$$

then the continuous representation of equation (3) is equivalent to

$$y(x,u)=\sum_{j=0}^{5}\frac{\det(\Psi_j)}{\det(\Psi)}P_j(x) \quad (7)$$

where

$V=(y_n,y_{n+v1},y_{n+1},y_{n+v2},f_{n+2},g_{n+2})^T$, $P(x)=(P_0(x),P_1(x),P_2(x),P_3(x),P_4(x),P_5(x)\ )^T$,

$P_j(x)=\{1,x,x^2,x^3,\sin\omega x\,,\cos\omega x\}$

$$\Psi=\begin{bmatrix} P_0(x_n) & P_1(x_n) & P_2(x_n) & P_3(x_n) & P_4(x_n) & P_5(x_n) \\ P_0(x_{n+v1}) & P_1(x_{n+v1}) & P_2(x_{n+v1}) & P_3(x_{n+v1}) & P_4(x_{n+v1}) & P_5(x_{n+v1}) \\ P_0(x_{n+1}) & P_1(x_{n+1}) & P_2(x_{n+1}) & P_3(x_{n+1}) & P_4(x_{n+1}) & P_5(x_{n+1}) \\ P_0(x_{n+v2}) & P_0(x_{n+v2}) & P_0(x_{n+v2}) & P_0(x_{n+v2}) & P_0(x_{n+v2}) & P_0(x_{n+v2}) \\ P_0'(x_{n+2}) & P_1'(x_{n+2}) & P_2'(x_{n+2}) & P_3'(x_{n+2}) & P_4'(x_{n+2}) & P_5'(x_{n+2}) \\ P_0''(x_{n+2}) & P_1''(x_{n+2}) & P_2''(x_{n+2}) & P_3''(x_{n+2}) & P_4''(x_{n+2}) & P_5''(x_{n+2}) \end{bmatrix}$$

and Ψ_i is obtained by replacing the ith column of Ψ by V.

***Proof* (See Appendix)**

For emphasis, we note that equation (7) is the CBHTM given by

$$y(x,u) = \sum_{j=0}^{1} \alpha_j(x,u) y_{n+j} + \sum_{m=1}^{2} \alpha_{vm}(x,u)\, y_{n+vm} + h\beta_2(x,u) f_{n+2} + h^2 \delta_k(x,u) g_{n+2} \tag{8}$$

Differentiating equation (8) with respect to x once, we obtain

$$y'(x,u) = \frac{1}{h} \sum_{j=0}^{1} \alpha_j'(x,u) y_{n+j} + \sum_{j=1}^{2} \alpha_{vm}'(x,u) y_{n+vm} + \beta_2'(x,u) f_{n+2} + h\delta_2'(x,u) \tag{9}$$

Evaluating equation (8) at $x = x_{n+2}$ gives the discrete method $y_{n+2} = \mathrm{y}(x_n + 2h)$ which takes the form of the main method. Evaluating equation (9) at $x = x_{n+v1}$, $x = x_{n+1}$ and $x = x_{n+v2}$ respectively, give the complementary methods. The BHTM whose coefficients are in trigonometric form is presented in equations (10)-(13) while the corresponding converted series form are given by equations (14)-(17) respectively. According to Lambert (1973) and Duxbury (1999), series form of the coefficients is used to avoid significant losses in evaluating the coefficients that may occur as $u \to 0$.

$$\begin{aligned} y_{n+2} = {} & \alpha_0(\sin u, \cos u) y_n + \alpha_{v1}(\sin u, \cos u) y_{n+v1} + \alpha_1(\sin u, \cos u) y_{n+1} \\ & + \alpha_{v2}(\sin u, \cos u) y_{n+v2} + h\beta_2(x,u) f_{n+2} + h^2 \delta_2(x,u) g_{n+2} \end{aligned} \tag{10}$$

$$\begin{aligned} f_{n+v1} = {} & \frac{1}{h}\alpha_{0,1}(\sin u, \cos u) y_n + \frac{1}{h}\alpha_{v1,1}(\sin u, \cos u) y_{n+v1} + \frac{1}{h}\alpha_{1,1}(\sin u, \cos u) y_{n+1} \\ & + \frac{1}{h}\alpha_{v2,1}(\sin u, \cos u) y_{n+v2} + \beta_{2,1}(x,u) f_{n+2} + h\delta_{2,1}(x,u) g_{n+2} \end{aligned} \tag{11}$$

$$\begin{aligned} f_{n+1} = {} & \frac{1}{h}\alpha_{0,2}(\sin u, \cos u) y_n + \frac{1}{h}\alpha_{v1,2}(\sin u, \cos u) y_{n+v1} + \frac{1}{h}\alpha_{1,2}(\sin u, \cos u) y_{n+1} \\ & + \frac{1}{h}\alpha_{v2,2}(\sin u, \cos u) y_{n+v2} + \beta_{2,2}(x,u) f_{n+2} + h\delta_{2,2}(x,u) g_{n+2} \end{aligned} \tag{12}$$

$$\begin{aligned} f_{n+\frac{3}{2}} = {} & \frac{1}{h}\alpha_{0,3}(\sin u, \cos u) y_n + \frac{1}{h}\alpha_{v1,3}(\sin u, \cos u) y_{n+v1} + \frac{1}{h}\alpha_{1,3}(\sin u, \cos u) y_{n+1} \\ & + \frac{1}{h}\alpha_{v2,3}(\sin u, \cos u) y_{n+v2} + \beta_{2,3}(x,u) f_{n+2} + h\delta_{2,3}(x,u) g_{n+2} \end{aligned} \tag{13}$$

$$
\left.
\begin{aligned}
\alpha_0 &= -\frac{9}{415} - \frac{6852}{6027875}u^2 - \frac{11976137}{300188175000}u^4 - \frac{198290254567}{164443082265000000}u^6 + \ldots \\
\alpha_{v1} &= \frac{64}{415} + \frac{2376}{861125}u^2 + \frac{842951}{7147337500}u^4 + \frac{103395779387}{27407180377500000}u^6 + \ldots \\
\alpha_1 &= -\frac{216}{415} - \frac{108}{6027875}u^2 - \frac{7905041}{100062725000}u^4 - \frac{1212485182217}{383700525285000000}u^6 + \ldots \\
\alpha_{v2} &= \frac{576}{415} - \frac{9672}{6027875}u^2 + \frac{143659}{150094087500}u^4 + \frac{341432297183}{575550787927500000}u^6 + \ldots \\
\beta_2 &= \frac{30}{83}h + \frac{36}{34445}h\,u^2 + \frac{74429}{4002509000}h\,u^4 + \frac{5891297173}{15348021011400000}h\,u^6 + \ldots \\
\gamma_2 &= -\frac{18}{415}h^2 - \frac{3744}{6027875}h^2u^2 - \frac{675919}{50031362500}h^2u^4 - \frac{10444925363}{31975043773750000}h^2u^6 + \ldots
\end{aligned}
\right\} \tag{14}
$$

$$
\left.
\begin{aligned}
\alpha_{0,1} &= -\frac{333}{830\,h} - \frac{1135527}{96446000}\frac{u^2}{h} - \frac{2296275749}{9606021600000}\frac{u^4}{h} - \frac{175095954228763}{36835250427360000000}\frac{u^6}{h} \\
&\quad + \ldots \\
\alpha_{v1,1} &= -\frac{891}{415\,h} + \frac{36486}{861125}\frac{u^2}{h} + \frac{96912601}{114357400000}\frac{u^4}{h} + \frac{32097930863}{1957655741250000}\frac{u^6}{h} + \ldots \\
\alpha_{1,1} &= \frac{3213}{830\,h} - \frac{4958433}{96446000}\frac{u^2}{h} - \frac{3190083257}{3202007200000}\frac{u^4}{h} - \frac{222465534698159}{12278416809120000000}\frac{u^6}{h} \\
&\quad + \ldots \\
\alpha_{v2,1} &= -\frac{549}{415\,h} + \frac{250941}{12055750}\frac{u^2}{h} + \frac{931466759}{2401505400000}\frac{u^4}{h} + \frac{29817236400629}{4604406303420000000}\frac{u^6}{h} \\
&\quad + \ldots \\
\beta_{2,1} &= \frac{31}{166} - \frac{549}{551120}u^2 - \frac{1185067}{128080288000}u^4 + \frac{101499116371}{491136672364800000}u^6 + \ldots \\
\gamma_{2,1} &= -\frac{87}{1660}h - \frac{195813}{192892000}h\,u^2 - \frac{164198527}{6404014400000}h\,u^4 - \frac{5639116978633}{8185611206080000000}h\,u^6 \\
&\quad + \ldots
\end{aligned}
\right\} \tag{15}
$$

$$
\left.
\begin{aligned}
\alpha_{0,2} &= \frac{127}{1245} + \frac{86591}{18083625}u^2 + \frac{1058121}{7147337500}u^4 + \frac{6123634502537}{1534802101140000000}u^6 + \ldots \\
\alpha_{v1,2} &= -\frac{424}{415} - \frac{40583}{2583375}u^2 - \frac{253731767}{514608300000}u^4 - \frac{26177549476279}{1973316987180000000}u^6 + \ldots \\
\alpha_{1,2} &= -\frac{229}{415} + \frac{275839}{18083625}u^2 + \frac{898694543}{1801129050000}u^4 + \frac{184601891677757}{13813218910260000000}u^6 + \ldots \\
\alpha_{v2,2} &= \frac{1832}{1245} - \frac{78349}{18083625}u^2 - \frac{554559701}{3602258100000}u^4 - \frac{56471755866637}{13813218910260000000}u^6 + \ldots \\
\beta_{2,2} &= -\frac{12}{83}h - \frac{31}{34445}h\,u^2 - \frac{516589}{24015054000}h\,u^4 - \frac{110304462811}{184176252136800000}h\,u^6 + \ldots \\
\gamma_{2,2} &= \frac{31}{830}h^2 + \frac{73133}{72334500}h^2u^2 + \frac{408708467}{14409032400000}h^2u^4 \\
&\quad + \frac{42723218897929}{55252875641040000000}h^2u^6 + \ldots
\end{aligned}
\right\} \tag{16}
$$

$$\left.\begin{aligned}
\alpha_{0,3} &= -\frac{187}{2490} - \frac{1157353}{289338000}u^2 - \frac{4048879811}{28818064800000}u^4 - \frac{155204307151319}{368352504273600000000}u^6 + \ldots \\
\alpha_{v1,3} &= \frac{237}{415} + \frac{19361}{1722250}u^2 + \frac{49499863}{114357400000}u^4 + \frac{980424049901}{73085814340000000}u^6 + \ldots \\
\alpha_{1,3} &= -\frac{1911}{830} - \frac{555029}{96446000}u^2 - \frac{1099105941}{3202007200000}u^4 - \frac{48723895228089}{4092805603040000000}u^6 + \ldots \\
\alpha_{v2,3} &= \frac{2249}{1245} - \frac{26888}{18083625}u^2 + \frac{366716951}{7204516200000}u^4 + \frac{222289381817}{8222154113250000}u^6 + \ldots \\
\beta_{2,3} &= \frac{51}{166}h + \frac{1303}{551120}hu^2 + \frac{19395587}{384240864000}hu^4 + \frac{560929758623}{4911366723648000000}hu^6 + \ldots \\
\gamma_{2,3} &= -\frac{111}{1660}h^2 - \frac{305469}{192892000}h^2u^2 - \frac{781888853}{19212043200000}h^2u^4 \\
&\quad - \frac{25788332564387}{2455683361824000000}h^2u^6 + \ldots
\end{aligned}\right\} \tag{17}$$

3. Analysis of BTHM

3.1 Local Truncation Error of BHTM

Theorem 2

The BHTM has a local truncation error (LTE) of $C_6h^6\left(\omega^2y^4(x_n) + y^{(6)}(x_n)\right) + O\left(h^{(7)}\right)$

Proof:

Consider the Taylor series expansion of the following

$y_n, y_{n+1}, y_{n+v1}, y_{n+v2}, y(x_n), y(x_n+h), y(x_n+v1h), y(x_n+v2h), y'_{n+2}, y'(x_n+2h), y''_{n+2}$ and $y''(x_n+2h)$. Also, assume that $y(x_{n+j}) = y_{n+j}, j = 0,1,v1,v2,$ $y'(x_{n+2}) = f_{n+2}$ and $y''(x_{n+2}) = g_{n+2}$. Then by substituting these into method in equation (10) and simplifying, we have that

$$LTE = y(x_{n+2}) - y_{n+2}$$

$$= C_6h^6\left(\omega^2y^{(4)}(x_n) + y^{(6)}(x_n)\right) + O\left(h^{(7)}\right) \quad \blacksquare \tag{18}$$

Consequently, the Local Truncation Error (LTE) of equations (10)-(13) are respectively obtained as

$$LTE = \begin{bmatrix} \left(-\frac{1551}{106240}y^{(6)}(x_n)+\frac{531}{531200}\omega^2 y^{(4)}(x_n)\right)h^6+O(h^7) \\ \left(\frac{73}{3320}y^{(6)}(x_n)-\frac{601}{1195200}\omega^2 y^{(4)}(x_n)\right)h^6+O(h^7) \\ \left(\frac{551}{21248}y^{(6)}(x_n)+\frac{859}{1593600}\omega^2 y^{(4)}(x_n)\right)h^6+O(h^7) \\ \left(\frac{16}{747}y^{(6)}(x_n)+\frac{3}{16600}\omega^2 y^{(4)}(x_n)\right)h^6+O(h^7) \end{bmatrix} \tag{19}$$

From equation (24), the order of BHTM is $p=(5,5,5,5)^T$ with error constants $C_6=\left(\frac{531}{531200},-\frac{601}{1195200},\frac{859}{1593600},\frac{3}{16600}\right)^T$. Also, following the definition of Lambert (1973) and Fatunla (1988), a numerical method is consistent if its order is greater than one. We therefore remark that BHTM is consistent.

3.2 Stability of BHTM

Following Akinfenwa *et al.* (2015), BHTM can be represented by a block matrix finite difference equation given by

$$A^{(1)}Y_{w+1}=A^{(0)}Y_w+hB^{(0)}F_w+hB^{(1)}F_{w+1}+h^2G_{w+1}D^{(1)} \tag{20}$$

where

$Y_{w+1}=(y_{n+v1},y_{n+1},y_{n+v2},y_{n+2})^T, Y_w=(y_{n-v2},y_{n-1},y_{n-v1},y_n)^T, F_{w+1}=$ $(f_{n+v1},f_{n+1},f_{n+v2},f_{n+2})^T, F_w=(f_{n-v2},f_{n-1},f_{n-v1},f_n)^T, G_{w+1}=(g_{n+v1},g_{n+1},,g_{n+v2},g_{n+2})^T$ and $A^{(0)},A^{(1)},B^{(0)},B^{(1)},D^{(1)}$ are $k\times k$ matrices.

For $k=2$, we have

$$A^{(1)}=\begin{bmatrix}\alpha_{v1,1} & \alpha_{1,1} & \alpha_{v2,1} & 0\\ \alpha_{v1,2} & \alpha_{1,2} & \alpha_{v2,2} & 0\\ \alpha_{v1,3} & \alpha_{1,3} & \alpha_{v2,3} & 0\\ \alpha_{v1} & \alpha_1 & \alpha_{v2} & 1\end{bmatrix},\ A^{(0)}=\begin{bmatrix}0&0&0&\alpha_{0,1}\\0&0&0&\alpha_{0,2}\\0&0&0&\alpha_{0,3}\\0&0&0&\alpha_0\end{bmatrix},\ B^{(1)}=\begin{bmatrix}1&0&0&\beta_{2,1}\\0&1&0&\beta_{2,2}\\0&0&1&\beta_{2,3}\\0&0&0&\beta_2\end{bmatrix},$$

$$B^{(0)}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},D^{(1)}=\begin{bmatrix}0&0&0&\gamma_{2,1}\\0&0&0&\gamma_{2,2}\\0&0&0&\gamma_{2,3}\\0&0&0&\gamma_2\end{bmatrix}$$

3.2.1 Zero Stability

According to Lambert (1973) and Fatunla (1988), BHTM is zero stable if the roots of the first characteristic polynomial have modulus less than or equal to one and those of modulus one are simple. i.e.

$\rho(R) = \det[RA^{(1)} - A^{(0)}] = 0$ and $|R_i| \leq 1$. Following this definition we obtained from our calculation that $|R| = (0,0,0,1)$. Since $|R_i| \leq 1$ *and* $|R_i| = 1$ is simple, BHTM is Zero Stable.

3.2.2 Convergence of BHTM

The necessary and sufficient condition for a method to be convergent is that it must be zero stable and consistent (Lambert, 1973 and Fatunla, 1988). Since BHTM is both zero stable and consistent, we therefore remark that it is convergent.

3.2.3 Linear Stability and Region of Absolute Stability of BHTM

Applying the block method to the test equations $y' = \lambda y$ and $y" = \lambda^2 y$ and letting $z = \lambda h$ yields $Y_{w+1} = \sigma(z,u)Y_w$, where $\sigma(z,u) = \frac{A^{(0)}+zB^{(0)}}{A^{(1)}-zB^{(0)}-z^2D^{(1)}}$. The matrix $\sigma(z)$ for BHTM has eigenvalues given by $(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = (0,0,0,\varphi_4)$, where $\varphi_4(z,u) = \frac{\eta_4(z,u)}{\tau_4(z,u)}$ is called the stability function. According to Ndukum *et al.* (2017), having suitable values of u in a large interval means that the method can cope well for problems with estimated frequencies. It is observed that for BHTM, the values of $u \epsilon [\pi, 2\pi)$ are satisfactory. The region of absolute stability (RAS) of BHTM is plotted for $u = \frac{\pi}{4}$ and is presented in figure 1.

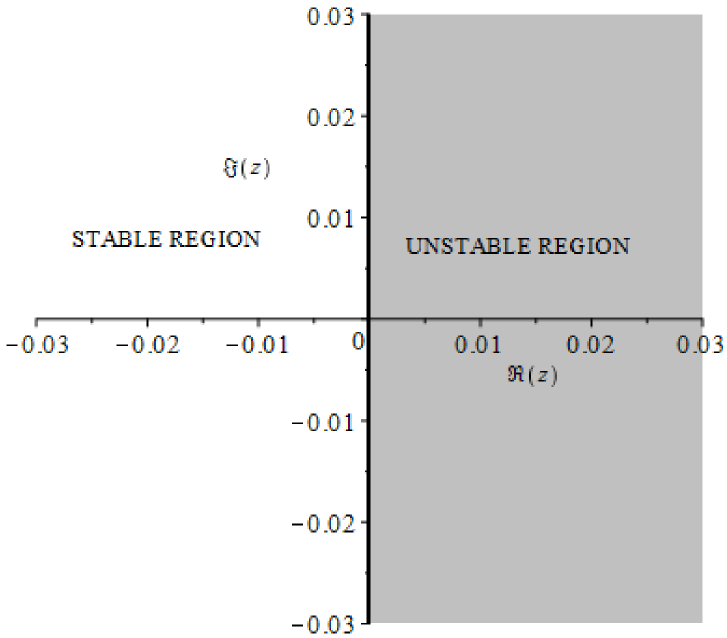

Figure 1. Region of Absolute Stability of BHTM

Definition: A-stability

A block method is said to be A-stable if its region of absolute stability contains the whole of the left plane.

Since the region of absolute stability of BHTM contains the whole of the left plane, then it is A stable.

4. Numerical Examples

We considered five nonlinear oscillatory problems to test the efficiency of the method and compare the results with results of some other methods established in literature.

Problem 1: Non Linear Perturbed Systems

Consider the nonlinear perturbed system on the range $[0, 10]$ with $\epsilon = 10^{-3}$.

$$y_1'' = \epsilon\varphi_1(x) - 25y_1 - \epsilon(y_1^2 + y_2^2) \qquad y_1(0) = 1 \;, \; y_1'(0) = 0$$

$$y_2'' = \epsilon\varphi_2(x) - 25y_2 - \epsilon(y_1^2 + y_2^2) \qquad y_2(0) = \epsilon \;, \; y_2'(0) = 5$$

where

$$\varphi_1(x) = 1 + \epsilon^2 + 2\epsilon\sin(5x + x^2) + 2\cos(x^2) + (25 - 4x^2)\sin(x^2)$$

$$\varphi_2(x) = 1 + \epsilon^2 + 2\epsilon\sin(5x + x^2) - 2\sin(x^2) + (25 - 4x^2)\cos(x^2)$$

The exact solution is given by

$$y_1(x) = \cos(5x) + \epsilon\sin(x^2) \;, \; y_2(x) = \sin(5x) + \epsilon\cos(x^2)$$

which according for Fang *et al.* (2009) represents a periodic motion of constant frequency with small perturbation of variable frequency. As selected by Fang *et al.* (2009) and Ngwane and Jator (2015a), we choose $\omega = 5$ and the numerical results of the maximum global errors of BHTM were compared with Block Hybrid Trigonometrically-Fitted (BHT) of Ngwane and Jator (2015a) and Trigonometrically-Fitted Adapted Runge-Kutta-Nyström (TFARKN 5(3)) of Fang et al. (2009) of order 5 each as presented in Table 1.

Table 1. Comparison of log of Maximum Errors and Number Steps

BHTM		BHT		TFARKN 5(3)	
N	$-\log_{10}(Err)$	N	$-\log_{10}(Err)$	$N(rejected)$	$-\log_{10}(Err)$
50	9.53	50	3.42	29(6)	2.78
100	10.39	100	4.61	88(9)	5.33
260	12.49	260	7.52	262(8)	7.85
810	15.82	810	10.43	811(4)	10.38

From Table 1 and Figure 2 it can be seen that BHTM outperformed BHT which is implemented in a corresponding fixed step size mode and TFARKN 5(3) which is implemented in variable step size mode respectively.

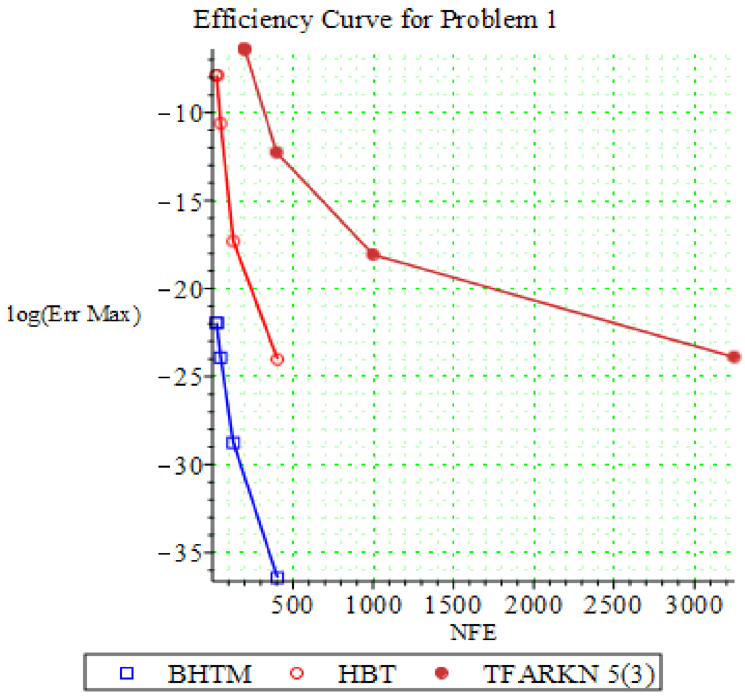

Figure 2. Efficiency curve for Problem 1

Problem 2: Nonlinear Strehmel-Weiner problem

We consider the nonlinear second order IVP which was also solved by Nguyen et al. (2007) and Jator (2016) in the interval $0 \leq t \leq 10$ respectively

$$y_1''(t) = (y_1(t) - y_2(t))^3 + 6368y_1(t) - 6384y_2(t) + 42\cos(10t), \quad y_1(0) = 0.5, y_1'(0) = 0$$
$$y_1''(t) = -(y_1(t) - y_2(t))^3 + 12768y_1(t) - 12784y_2(t) + 42\cos(10t), \quad y_2(0) = 0.5, y_2'(0) = 0$$

with solution in closed form given by $y_1(t) = y_2(t) = \cos(4t) - \frac{\cos(10t)}{2}$.

Numerical results of the maximum global errors of BHTM were compared with order 6 Symmetric Boundary Value Method (SBVM) of Jator (2016) and Trigonometric Implicit Runge-Kutta Methods

(TIRKM) of Nguyen (2007) are presented in Table 2.

Table 2: Comparison of Maximum Errors

$BHTM$		$SVBM$		$TIRKM$	
NFE	Err	NFE	Err	NFE	Err
600	6.02×10^{-7}	801	2.6×10^{-7}	907	2.5×10^{-4}
900	7.90×10^{-8}	1201	1.6×10^{-8}	1288	6.6×10^{-6}
1600	8.11×10^{-9}	1601	2.8×10^{-9}	1682	7.0×10^{-6}

In Table 2 and Figure 3, we show that the BHTM uses fewer number of function evaluation and hence more efficient than the order 6 methods in Jator (2016) and Nguyen (2007) respectively.

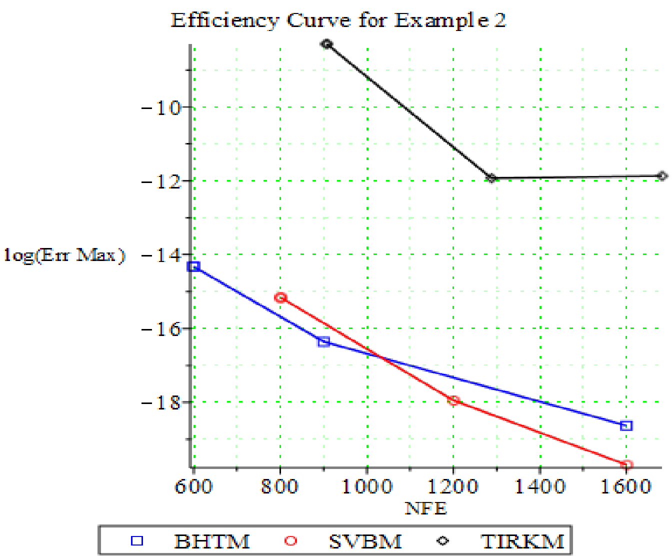

Figure 3: Efficiency curve for Problem 2

Problem 3: Two Body Problem

We consider the following system of coupled differential equations which is well known as the two body problem:

$$y_1''(x) = -\frac{y_1}{r^3}, \quad y_1(0) = 1, y_1'(0) = 0$$

$$y_2''(x) = -\frac{y_2}{r^3}, \quad y_2(0) = 0, y_2'(0) = 1$$

where $r = \sqrt{y_1^2 + y_2^2}$ and whose analytical solution is given by

$y_1(x) = \cos(x), y_2(x) = \sin(x)$

The problem was considered in Senu *et al.* (2010) in the interval $0 \leq x \leq 10$ with $\omega = 1$. The BHTM is compared with the fourth order DIRKNNew of Senu *et al.* (2010) and the numerical results are displayed in Table 3.

Table 3. Comparison of Numerical Results

BHTM		DIRKNNew	
NFE	Err	NFE	Err
1600	5.13×10^{-27}	5000	7.49×10^{-4}
3200	1.30×10^{-29}	10000	5.62×10^{-7}
6400	4.00×10^{-30}	25000	1.00×10^{-9}
12800	7.43×10^{-28}	60000	1.78×10^{-7}

Table 3 reveals that BHTM gives better approximation. Also, Figure 4 shows that the method in this paper is more efficient

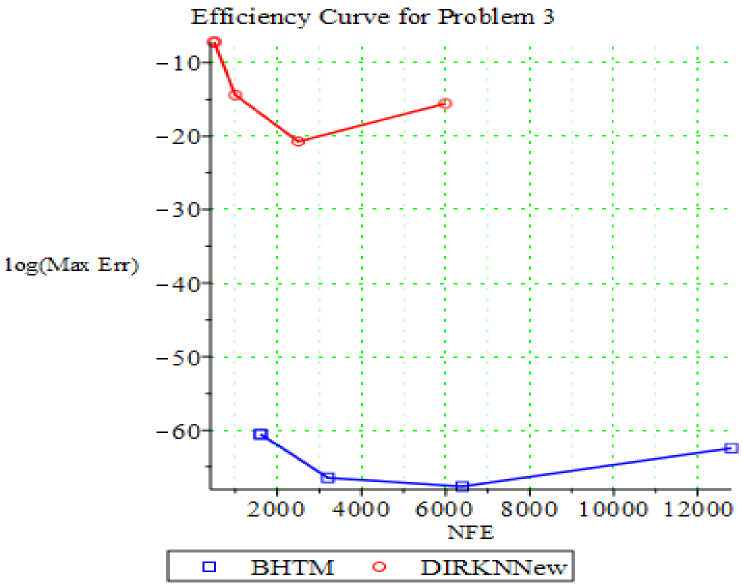

Figure 4: Efficiency curve for Problem 3

Problem 4: Undamped Duffing Equation

Consider the periodically forced nonlinear IVP that was solved in Fang *et al.* (2009).

$$\begin{cases} y'' = (\cos(t) + \epsilon \sin(10t))^3 - 99\epsilon \sin(10t) - y^3 - y, & 0 \leq t \leq 1000 \\ y(0) = 1, \quad y'(0) = 10\epsilon \end{cases}$$

with $\epsilon = 10^{-10}$ and whose analytic solution $y(t) = \cos(t) + \epsilon \sin(10t)$ describes a periodic motion of low frequency with a small perturbation of high frequency. In this problem, $\omega = 1$ is selected and the numerical results in comparison with TFARKN 5(3) of order 5 are displayed in Table 4 while the efficiency curve is presented in Figure 5.

Table 4. Comparison of Numerical results

BHTM			TFARKN 5(3)		
N	$Max\ Error$	NFE	$N\ (rejected)$	$Max\ Error$	NFE
80	2.19×10^{-17}	240	80 (28)	2.63×10^{-2}	300
97	8.37×10^{-18}	294	97 (41)	4.47×10^{-6}	400
135	1.61×10^{-18}	378	135 (36)	3.72×10^{-8}	600
1035	6.13×10^{-23}	2108	1035 (268)	1.17×10^{-13}	4200

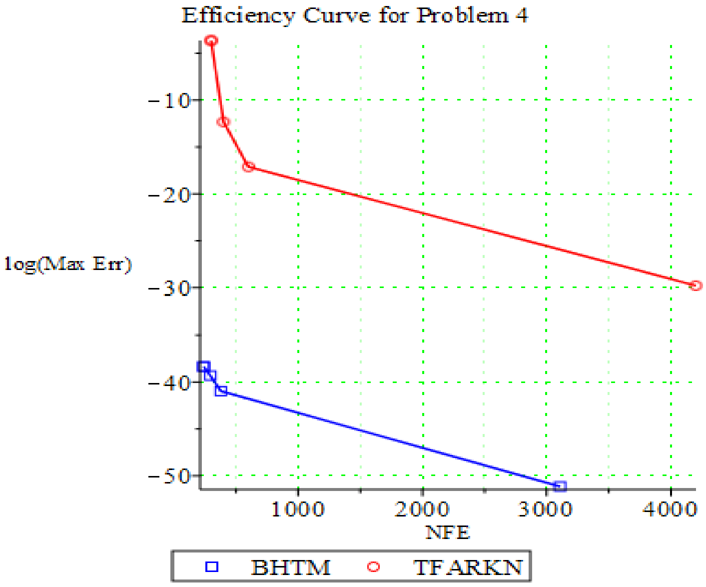

Figure 5: Comparison of Efficiency curve

Problem 5: Nonlinear Duffing Equation

Consider the nonlinear Duffing equation forced by a harmonic function given by $y'' + y + y^3 = B\cos(\Omega x)$ whose theoretical solution is unknown. A very accurate approximation of the theoretical solution of this equation is judge by comparison with a Galerkin approximation obtained by Van Dooren (1974) given by $y(x) = C_1 \cos(\Omega x) + C_2 \cos(3\Omega x) + C_3 \cos(5\Omega x) + C_4 \cos(7\Omega x)$ and the appropriate initial conditions are $y(0) = C_0 \quad y'(0) = 0$

where

$\Omega = 1.01$, $B = 0.002$, $C_0 = 0.200426728069$ $C_1 = 0.200179477536$, $C_2 = 0.246946143 \times 10^{-3}$, $C_3 = 0.304016 \times 10^{-6}$, $C_4 = 0.374 \times 10^{-9}$

This problem has been solved numerically by different researchers in the literature. Simos (1993), Wang *et al.* (2005)and Van Daele and Van Berghe (2007) all solved the problem with P Stable Obrechkoff methods of order 12 in the interval $\left[0, \frac{40.5\pi}{1.01}\right]$. Archar (2011) also considered the problem for an order 12 symmetric Obrechkoff methods within the same interval of integrations. The newly developed BHTM is compared with the aforementioned methods and the end point absolute errors at $x = \frac{40.5\pi}{1.01}$ and the CPU time are displayed in Tables 5 and 6 respectively.

Table 5. Comparison of Error

h	Simos	Wang et al.	Daele	Achar	BHTM
$\frac{m}{500}$	$3.15E-4$	$4.08E-5$	$4.06E-5$	$4.09E-5$	$1.84E-7$
$\frac{m}{1000}$	$1.81E-5$	$1.27E-6$	$1.87E-6$	$1.27E-6$	$6.49E-9$
$\frac{m}{2000}$	$1.08E-6$	$3.93E-8$	$3.84E-8$	$3.94E-8$	$1.73E-10$
$\frac{m}{3000}$	$2.09E-7$	$5.17E-9$	$5.13E-9$	$5.18E-9$	$2.32E-11$
$\frac{m}{4000}$	$6.55E-8$	$1.23E-9$	$3.19E-9$	$1.23E-9$	$6.95E-12$
$\frac{m}{5000}$	$2.67E-8$	$4.07E-10$	$9.89E-10$	$4.09E-10$	$3.69E-12$

Table 6: Comparison of CPU Time

h	Simos	Wang et al.	Daele	Achar	BHTM
$\frac{m}{500}$	1.437	1.406	1.484	1.188	1.0254
$\frac{m}{1000}$	2.892	2.891	2.938	2.312	0.7198
$\frac{m}{2000}$	6.233	6.236	6.360	4.812	1.9776
$\frac{m}{3000}$	9.859	9.546	9.719	7.548	2.3468

$\frac{m}{4000}$	13.548	13.063	13.390	9.986	2.4463
$\frac{m}{5000}$	16.922	16.499	16.969	12.860	2.9522

General speaking, higher order methods are expected to be more accurate than the lower order methods. However in Table 5 the reverse is the case as BHTM of order 5 is more accurate for this problem than some of the existing higher order methods in the literature. The Efficiency Curve is displayed in the Figure 6.

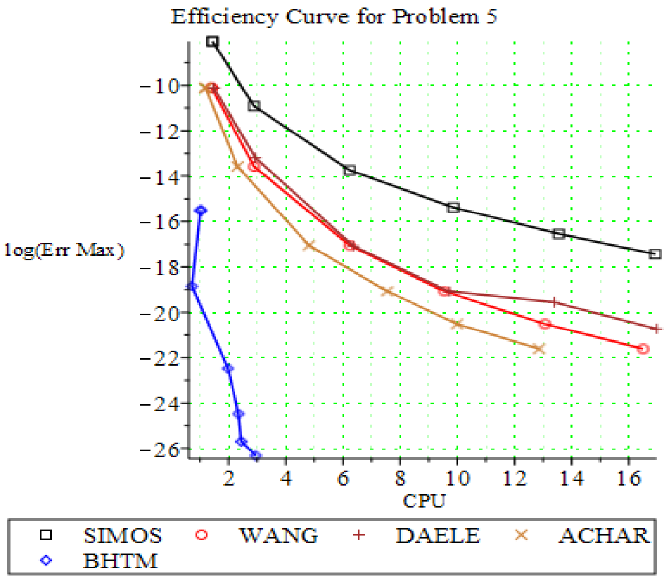

Figure 6: Comparison of Efficiency curve

5.0 Conclusion

A new block hybrid trigonometrically-fitted method of order five is constructed via multistep collocation technique in this paper. The stability properties of the method was analyzed and was found to be A-stable. Numerical results on representative examples of homogenous, non-homogenous and autonomous nonlinear oscillatory problems are reported to show the robustness and competence of the new method compared to some efficient methods in the literature. We therefore conclude that for second order nonlinear initial value problems with oscillatory solution, BHTM is more accurate and more efficient than some of the existing methods in the literature.

Conflict of Interest

The authors declare that they have no conflict of interest.

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APPENDICES

1. Coefficients of Main method of BHTM in Trigonometric form

$$\alpha_0 = \left(18u^2\sin\left(\frac{1}{2}u\right) - 9u^2\sin(u) + 2u^2\sin\left(\frac{3}{2}u\right) - 3u^3 + 60u\cos\left(\frac{1}{2}u\right) - 48u\cos(u) + 12\cos\left(\frac{3}{2}u\right)u - 188\sin\left(\frac{1}{2}u\right) + 136\sin(u) - 28\sin\left(\frac{3}{2}u\right) - 24u\right) \Big/ \left(-24\cos(2u)u - 96u\cos(u) - 57u^2\sin(u) - 11\sin(2u)u^2 + 36u\cos\left(\frac{1}{2}u\right) + 84\cos\left(\frac{3}{2}u\right)u + 26u^2\sin\left(\frac{1}{2}u\right) + 42u^2\sin\left(\frac{3}{2}u\right) + 76\sin\left(\frac{3}{2}u\right) + 428\sin\left(\frac{1}{2}u\right) - 328\sin(u) + 3u^3\right)$$

$$\alpha_{v1} = \left(-64u^2\sin\left(\frac{1}{2}u\right) - 2\sin(2u)u^2 + 24u^2\sin(u) + 12u^3 - 192u\cos\left(\frac{1}{2}u\right) - 12\cos(2u)u + 120u\cos(u) + 480\sin\left(\frac{1}{2}u\right) + 28\sin(2u) - 248\sin(u) - 32\sin\left(\frac{3}{2}u\right) + 84u\right) \Big/ \left(-24\cos(2u)u - 96u\cos(u) - 57u^2\sin(u) - 11\sin(2u)u^2 + 36u\cos\left(\frac{1}{2}u\right) + 84\cos\left(\frac{3}{2}u\right)u + 26u^2\sin\left(\frac{1}{2}u\right) + 42u^2\sin\left(\frac{3}{2}u\right) + 76\sin\left(\frac{3}{2}u\right) + 428\sin\left(\frac{1}{2}u\right) - 328\sin(u) + 3u^3\right)$$

$$\alpha_1 = \left(72u^2\sin\left(\frac{1}{2}u\right) + 9\sin(2u)u^2 - 24u^2\sin\left(\frac{3}{2}u\right) - 18u^3 + 168u\cos\left(\frac{1}{2}u\right) + 48\cos(2u)u - 120\cos\left(\frac{3}{2}u\right)u - 152\sin\left(\frac{1}{2}u\right) - 104\sin(2u) - 256\sin(u) + 360\sin\left(\frac{3}{2}u\right) - 96u\right) \Big/ \left(-24\cos(2u)u - 96u\cos(u) - 57u^2\sin(u) - 11\sin(2u)u^2 + 36u\cos\left(\frac{1}{2}u\right) + 84\cos\left(\frac{3}{2}u\right)u + 26u^2\sin\left(\frac{1}{2}u\right) + 42u^2\sin\left(\frac{3}{2}u\right) + 76\sin\left(\frac{3}{2}u\right) + 428\sin\left(\frac{1}{2}u\right) - 328\sin(u) + 3u^3\right)$$

$$\alpha_{v2} = \left(-18\sin(2u)u^2 - 72u^2\sin(u) + 64u^2\sin\left(\frac{3}{2}u\right) + 12u^3 - 60\cos(2u)u - 168u\cos(u) + 192\cos\left(\frac{3}{2}u\right)u + 288\sin\left(\frac{1}{2}u\right) + 76\sin(2u) + 40\sin(u) - 224\sin\left(\frac{3}{2}u\right) + 36u\right) \Big/ \left(-24\cos(2u)u - 96u\cos(u) - 57u^2\sin(u) - 11\sin(2u)u^2 + 36u\cos\left(\frac{1}{2}u\right) + 84\cos\left(\frac{3}{2}u\right)u + 26u^2\sin\left(\frac{1}{2}u\right) + 42u^2\sin\left(\frac{3}{2}u\right) + 76\sin\left(\frac{3}{2}u\right) + 428\sin\left(\frac{1}{2}u\right) - 328\sin(u) + 3u^3\right)$$

$$\beta_2 = \left(12\sin\left(\frac{1}{2}u\right)hu^2 - 3\sin(2u)hu^2 - 18\sin(u)hu^2 + 12\sin\left(\frac{3}{2}u\right)hu^2 + 336\sin\left(\frac{1}{2}u\right)h - 24\sin(2u)h - 336\sin(u)h + 144\sin\left(\frac{3}{2}u\right)h\right) \Big/ \left(-24\cos(2u)u - 96u\cos(u) - 57u^2\sin(u) - 11\sin(2u)u^2 + 36u\cos\left(\frac{1}{2}u\right) + 84\cos\left(\frac{3}{2}u\right)u + 26u^2\sin\left(\frac{1}{2}u\right) + 42u^2\sin\left(\frac{3}{2}u\right) + 76\sin\left(\frac{3}{2}u\right) + 428\sin\left(\frac{1}{2}u\right) - 328\sin(u) + 3u^3\right)$$

$$\gamma_2 = \left(12\cos\left(\frac{1}{2}u\right)h^2u - 3\cos(2u)h^2u - 18\cos(u)h^2u + 12\cos\left(\frac{3}{2}u\right)h^2u - 124\sin\left(\frac{1}{2}u\right)h^2 + 11\sin(2u)h^2 + 130\sin(u)h^2 - 60\sin\left(\frac{3}{2}u\right)h^2 - 3h^2u\right) \Big/ \left(-24\cos(2u)u - 96u\cos(u) - 57u^2\sin(u) - 11\sin(2u)u^2 + 36u\cos\left(\frac{1}{2}u\right) + 84\cos\left(\frac{3}{2}u\right)u + 26u^2\sin\left(\frac{1}{2}u\right) + 42u^2\sin\left(\frac{3}{2}u\right) + 76\sin\left(\frac{3}{2}u\right) + 428\sin\left(\frac{1}{2}u\right) - 328\sin(u) + 3u^3\right)$$

2. Proof of Theorem 1

The proof of the theorem is in the spirit of Ngwane and Jator (2015b).

We require that equation (3) be defined by the assumed basis function as follows

$$\alpha_j(x,u) = \sum_{i=0}^{5} \alpha_{i,j}(x,u)\,P_i(x) \qquad j = 0,1 \tag{21}$$

$$\alpha_{vm}(x,u) = \sum_{i=0}^{5} \alpha_{i,vm}(x,u)\,P_i(x) \qquad m = 1,2 \tag{22}$$

$$h\beta_j(x,u) = \sum_{i=0}^{5} h\beta_{i,2}(x,u)\,P_i(x) \tag{23}$$

$$h^2\gamma_2(x,u) = \sum_{i=0}^{5} h^2\gamma_{i,2}(x,u)\,P_i(x) \tag{24}$$

Substituting equations (21)-(24) into equation (3) yield

$$y(x,u) = \sum_{i=0}^{5}\left\{\sum_{j=0}^{1} \alpha_{i,j}(x,u)\,y_{n+j} + \sum_{m=1}^{2} \alpha_{i,vm}(x,u)\,y_{n+vm} + h\beta_{i,2}(x,u)f_{n+2} + h^2\gamma_{i,2}(x,u)g_{n+2}\right\}P_i(x) \tag{25}$$

Letting

$$\Lambda_i = \sum_{j=0}^{1} \alpha_{i,j}(x,u)\, y_{n+j} + \sum_{m=1}^{2} \alpha_{i,vm}(x,u)\, y_{n+vm} + h\beta_{i,2}(x,u) f_{n+2} + h^2 \gamma_{i,2}(x,u) g_{n+2}$$

equation (25) becomes

$$\sum_{i=0}^{5} \Lambda_i P_i(x) \qquad (26)$$

Imposing the conditions in equations (4)-(6) on equation (26), we obtain a system of 6 equations which is expressed as $\Psi\Lambda = \mathrm{V}$ where $\Lambda = (\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5)^T$ is a vector form of 6 undetermined coefficients that are determined by applying Crammer's rule as given in equation (27)

$$\Lambda_i = \frac{\det(\Psi_i)}{\det(\Psi)}, \qquad i = 0(1)(5) \qquad (27)$$

Ψ_i is obtained by replacing the ith column of Ψ by V.

Substituting equation (27) into equation (26) to obtain

$$y(x,u) = \sum_{i=0}^{5} \frac{\det(\Psi_i)}{\det(\Psi)} P_i(x) \qquad \blacksquare$$