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New Practical Determination of Non-Singular H-Matrices

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Abstract

The sufficient conditions for determining the non-singular H-matrices were given by applying the theory of diagonally dominant matrix. Then, the results of a recent study were promoted and improved with such conditions. Through the analysis of a numerical example, the proposed conditions were proved as more applicable than those in the reference.

Key words

Non-singular H-matrix, Diagonal dominance, Irreducibility, Chain of nonzero elements.

1. Introduction

H-matrices have been extensively applied in fields like cybernetics, power system, economic mathematics and elastic mechanics. However, it is very difficult to determine such matrices in real practice. Much research has been done on the determination of H-matrices. For instance, Reference [1] improved some existing determination results with a novel method. The determination conditions were updated here to further improve the main results in Reference.

2. Analysis

Let $M_N(c)$ be a set of complex matrices of order n, and $N=\{1,2,\ldots,n\}$.

It is assumed that

$$A=(a_{ij})\in M_n(c),\ \Lambda_i(A)=\sum_{j\neq i}|a_{ij}|.$$

$N_1 = \{i \in N : 0 < |a_{ii}| = \Lambda_i(A)\}$
$N_2 = \{i \in N : 0 < |a_{ii}| < \Lambda_i(A)\}$
$N_3 = \{i \in N : |a_{ii}| > \Lambda_i(A)\}$

It is clear that $N = N_1 \bigcup N_2 \bigcup N_3$ and $N_1 \bigcap N_2 \bigcap N_3 = \varnothing$.

Suppose $A = (a_{ij}) \in M_n(c)$. If $|a_{ii}| > \Lambda_i(A)$ and $\forall i \in N$, then A is a strictly diagonally dominant matrix and $A \in D$. If there exists a diagonally dominant positive-definite matrix X such that $AX \in D$, then A is a generalized strictly diagonally dominant matrix (i.e. A is a non-singular H-matrix), denoted as $A \in D$.

It is well known that if A is a non-singular H-matrix, there is at least a strictly diagonally dominant row, i.e. $N_3 = \varnothing$. Given that $N_1 \bigcap N_2 = \varnothing$, it is possible to derive that $A \in D$. Hence, it is assumed that $N_1 \bigcap N_2 = \varnothing$ and $N_3 = \varnothing$ in this research. The diagonal elements of the matrices here are set to nonzero because none of the main diagonal elements in a non-singular H-matrix is zero.

Definition 1

Assuming that $A = (a_{ij}) \in M_n(c)$ is an irreducible matrix satisfying $|a_{ii}| \geq \Lambda_i(A)(i \in N)$, and that at least one strict inequality holds, A is an irreducible diagonally dominant matrix.

Definition 2

Suppose $A = (a_{ij}) \in M_n(c)$ satisfies $|a_{ii}| \geq \Lambda_i(A)(i \in N)$, and at least one strict inequality holds. For the i that is true for each equation, if there exists a chain of nonzero elements $a_{ij_1} a_{j_1 j_2} \ldots a_{j_{k-1} j_k} \neq 0$ such that $|a_{j_k j_k}| > \Lambda_{j_k}(A)$, then A is a diagonally dominant matrix with a chain of nonzero elements.

Lemma 1

Assuming that $A = (a_{ij}) \in M_n(c)$, if A is an irreducible diagonally dominant matrix, then A is a non-singular H-matrix.

Lemma 2

Assuming that $A = (a_{ij}) \in M_n(c)$, if A is a diagonally dominant matrix with a chain of nonzero elements, then A is a non-singular H-matrix.

Reference gives the following definition when $A = (a_{ij}) \in M_n(c)$:

$$r = \max_{i \in N_3} \left(\frac{\sum_{t \in N_1} |a_{it}| + \sum_{t \in N_2} |a_{it}|}{|a_{ii}| - \sum_{\substack{t \in N_3 \\ t \neq i}} |a_{it}|} \right),$$

$P_i(A) = \sum_{t \in N_1} |a_{it}| + \sum_{t \in N_2} |a_{it}| + r \sum_{\substack{t \in N_3 \\ t \neq i}} |a_{it}|$, $(i \in N_3)$ and proposes the following theorem:

Assuming that $A = (a_{ij}) \in M_n(c)$, for any $i \in N_2$, there is

$$|a_{ii}| > \frac{\Lambda_i(A)}{\Lambda_i(A) - |a_{ii}|} \left[\sum_{t \in N_1} |a_{it}| + \sum_{t \in N_2} |a_{it}| \frac{\Lambda_t(A) - |a_{tt}|}{\Lambda_t(A)} + \sum_{t \in N_2} |a_{it}| \frac{P_t(A)}{|a_{tt}|} \right] \text{ and } |a_{ii}| \neq \sum_{\substack{t \in N_1 \\ t \neq i}} |a_{it}|, \ \forall i \in N_1,$$

then A is a non-singular H-matrix.

To improve the theorem, a new set of sufficient conditions were provided with a wider determination scope. For this purpose, it is defined that:

$$M = \max_{\substack{i \in N_2 \\ i \in N_3}} \left\{ \frac{\Lambda_i(A) - |a_{ii}|}{\Lambda_i(A)}, \frac{\Lambda_j(A)}{|a_{jj}|} \right\}$$

$$K = \max_{i \in 3} \left(\frac{M \sum_{t \in N_1} |a_{it}| + \sum_{t \in N_2} |a_{it}| \frac{\Lambda_t(A) - |a_{tt}|}{\Lambda_t(A)}}{|a_{ii}| - \sum_{\substack{t \in N_3 \\ t \neq i}} |a_{it}|} \right)$$

$$R_i(A) = M \sum_{t \in N_1} |a_{it}| + \sum_{t \in N_2} |a_{it}| \frac{\Lambda_t(A) - |a_{tt}|}{\Lambda_t(A)} + K \sum_{\substack{t \in N_3 \\ t \neq i}} |a_{it}| \ \ (i \in N_3).$$

By the definitions above, $0 < \mu < 1, k \leq r$. Since $\forall i \in N_3$, there is $R_i(A) \leq P_i(A)$

Theorem 1

Assuming that $A = (a_{ij}) \in M_n(c)$, and for any $i \in N_2$, there is

$$|a_{ii}| > \frac{\Lambda_i(A)}{\Lambda_i(A) - |a_{ii}|} \left[M \sum_{t \in N_1} |a_{it}| + \sum_{\substack{t \in N_2 \\ t \neq i}} |a_{it}| \frac{\Lambda_t(A) - |a_{tt}|}{\Lambda_t(A)} + \sum_{t \in N_3} |a_{it}| \frac{R_t(A)}{|a_{tt}|} \right] \tag{1}$$

and $|a_{ii}| \neq \sum_{\substack{t\in N_1 \\ t\neq i}} |a_{it}|, \forall i \in N_1$, then A is a non-singular H-matrix.

Proof:

By the definition of K, it is obtained that:

$$K|a_{ii}| \geq M\sum_{t\in N_1}|a_{it}| + \sum_{\substack{t\in N_2 \\ t\neq i}}|a_{it}|\frac{\Lambda_t(A)-|a_{tt}|}{\Lambda_t(A)} + \sum_{t\in N_3}|a_{it}| = R_i(A)$$

Hence

$$K \geq \frac{R_i(A)}{|a_{ii}|} \quad (i \in N_3) \tag{2}$$

The following can be derived based on equation (1):

$$|a_{ii}|\frac{\Lambda_i(A)-|a_{ii}|}{\Lambda_i(A)} - M\sum_{t\in N_1}|a_{it}| - \sum_{\substack{t\in N_2 \\ t\neq i}}|a_{it}|\frac{\Lambda_t(A)-|a_{tt}|}{\Lambda_t(A)} - \sum_{t\in N_3}|a_{it}|\frac{R_t(A)}{|a_{tt}|} > 0 \tag{3}$$

Whereas $R_i(A) < \Lambda_i(A)$ and $M \geq \frac{\Lambda_i(A)}{|a_{ii}|}$ under $\forall i \in N_3$, there is

$$M > \frac{R_i(A)}{|a_{ii}|} \tag{4}$$

Equations (3) and (4) demonstrate the existence of the sufficiently small

$$M > \frac{R_i(A)}{|a_{ii}|} + \varepsilon (i \in N_3) \tag{5}$$

$$|a_{ii}|\frac{\Lambda_i(A)-|a_{ii}|}{\Lambda_i(A)} - M\sum_{t\in N_1}|a_{it}| - \sum_{\substack{t\in N_2 \\ t\neq i}}|a_{it}|\frac{\Lambda_t(A)-|a_{ii}|}{\Lambda_t(A)} - \sum_{t\in N_3}|a_{it}|\frac{R_t(A)}{|a_{tt}|} > \varepsilon\sum_{t\in N_3}|a_{it}| \quad (i \in N_2) \tag{6}$$

Let us construct a diagonally dominant positive-definite matrix $X = diag(x_1, x_2, \ldots x_n)$, where

$$x_i = \begin{cases} M \quad , \qquad i \in N_1 \\ \dfrac{\Lambda_i(A) - |a_{ii}|}{\Lambda_i(A)}, i \in N_2 \\ \dfrac{R_i(A)}{|a_{ii}|} + \varepsilon \ , \quad i \in N_3 \end{cases}$$

Let $B = AX = (b_{ij})_{nxn}$.

Whereas $M \geq \dfrac{\Lambda_t(A) - |a_{tt}|}{\Lambda_t(A)}$ $(i \in N_2)$ and $M > \dfrac{R_i(A)}{|a_{ii}|} + \varepsilon$ $(i \in N_3)$, for $\forall i \in N_1$,

$$\Lambda_i(B) = M \sum_{\substack{t \in N_1 \\ t \neq i}} |a_{it}| + \sum_{t \in N_2} |a_{it}| \frac{\Lambda_t(A) - |a_{tt}|}{\Lambda_t(A)} + \sum_{t \in N_3} |a_{it}| \left(\frac{R_t(A)}{|a_{tt}|} + \varepsilon \right)$$

$$\leq M \sum_{\substack{t \in N_1 \\ t \neq i}} |a_{it}| + M \sum_{t \in N_2} |a_{it}| + M \sum_{t \in N_2} |a_{it}| = M \Lambda_i(A) = M |a_{ii}| = |b_{ii}|$$

According to equation (6), for $\forall i \in N_2$,

$$\Lambda_i(B) = M \sum_{\substack{t \in N_1 \\ t \neq i}} |a_{it}| + \sum_{t \in N_2} |a_{it}| \frac{\Lambda_t(A) - |a_{tt}|}{\Lambda_t(A)} + \sum_{t \in N_3} |a_{it}| \left(\frac{R_t(A)}{|a_{tt}|} + \varepsilon \right)$$

$$= M \sum_{t \in N_1} |a_{it}| + \sum_{\substack{t \in N_2 \\ t \neq i}} |a_{it}| \frac{\Lambda_t(A) - |a_{tt}|}{\Lambda_t(A)} + \sum_{t \in N_3} |a_{it}| \frac{R_t(A)}{|a_{tt}|} + \varepsilon \sum_{t \in N_3} |a_{it}|$$

$$< M \sum_{t \in N_1} |a_{it}| + \sum_{\substack{t \in N_2 \\ t \neq i}} |a_{it}| \frac{\Lambda_t(A) - |a_{tt}|}{\Lambda_t(A)} + \sum_{t \in N_3} |a_{it}| \frac{R_t(A)}{|a_{tt}|} + |a_{ii}| \frac{\Lambda_t(A) - |a_{tt}|}{\Lambda_t(A)}$$

$$-M \sum_{t \in N_1} |a_{it}| - \sum_{\substack{t \in N_2 \\ t \neq i}} |a_{it}| \frac{\Lambda_t(A) - |a_{tt}|}{\Lambda_t(A)} - \sum_{t \in N_3} |a_{it}| \frac{\Lambda_t(A)}{|a_{tt}|} = |a_{ii}| \frac{\Lambda_t(A) - |a_{tt}|}{\Lambda_t(A)} = |b_{ii}|$$

Whereas $K \geq \dfrac{R_i(A)}{|a_{ii}|}$, for $\forall i \in N_3$,

$$\Lambda_i(B)=M\sum_{t\in N_1}|a_{it}|+\sum_{t\in N_2}|a_{it}|\frac{\Lambda_t(A)-|a_{tt}|}{\Lambda_t(A)}+\sum_{\substack{t\in N_3\\t\neq i}}|a_{it}|\left(\frac{R_t(A)}{|a_{tt}|}+\varepsilon\right)$$

$$=M\sum_{t\in N_1}|a_{it}|+\sum_{t\in N_2}|a_{it}|\frac{\Lambda_t(A)-|a_{tt}|}{\Lambda_t(A)}+\sum_{\substack{t\in N_3\\t\neq i}}|a_{it}|\frac{R_t(A)}{|a_{tt}|}+\varepsilon\sum_{\substack{t\in N_3\\t\neq i}}|a_{it}|$$

$$=M\sum_{t\in N_1}|a_{it}|+\sum_{t\in N_2}|a_{it}|\frac{\Lambda_t(A)-|a_{tt}|}{\Lambda_t(A)}+\sum_{\substack{t\in N_3\\t\neq i}}|a_{it}|\frac{R_t(A)}{|a_{tt}|}+\varepsilon\sum_{\substack{t\in N_3\\t\neq i}}|a_{it}|$$

$$\leq M\sum_{t\in N_1}|a_{it}|+\sum_{t\in N_2}|a_{it}|\frac{\Lambda_t(A)-|a_{tt}|}{\Lambda_t(A)}+K\sum_{\substack{t\in N_3\\t\neq i}}|a_{it}|+\varepsilon\sum_{\substack{t\in N_3\\t\neq i}}|a_{it}|$$

$$=R_i(A)+\varepsilon\sum_{\substack{t\in N_3\\t\neq i}}|a_{it}|<R_i(A)+\varepsilon|a_{ii}|=|a_{ii}|\left(\frac{R_i(A)}{|a_{ii}|}+\varepsilon\right)=|b_{ii}|$$

By the theorem condition $|a_{ii}|\neq\sum_{\substack{t\in N_1\\t\neq i}}|a_{it}|$ $(\forall i\in N_1)$, for $\forall i\in N_1$, there exists $t\in N_2\bigcup N_3$ such that $a_{it}\neq 0$. In other words, for any vertex in N_1 that satisfies $|b_{ii}|=\Lambda_i(B)$, there is $b_{it}\neq 0$ such that $|b_{tt}|>\Lambda_t(B)$.

To sum up, B is a diagonally dominant matrix with a chain of nonzero elements. According to Lemma 2, B is a non-singular H-matrix, which, in turn, proves that A is a non-singular H-matrix.

Note: Under $0<M<1$ and $\forall i\in N_3$, there is: $0\leq\frac{R_i(A)}{|a_{ii}|}\leq\frac{P_i(A)}{|a_{ii}|}<1$,

Condition (1) of Theorem 1 contains the Condition (2.1) of Theorem 1 in Reference, indicating that the main results of Reference [1] have been improved. The improvement was validated by the numerical example in the subsequent section.

Theorem 2

Suppose $A=(a_{ij})\in M_n(c)$ is an irreducible matrix. If, for any $i\in N_2$, there is

$$|a_{ii}|>\frac{\Lambda_i(A)}{\Lambda_i(A)-|a_{ii}|}\left[M\sum_{t\in N_1}|a_{it}|+\sum_{\substack{t\in N_2\\t\neq i}}|a_{it}|\frac{\Lambda_t(A)-|a_{tt}|}{\Lambda_t(A)}+\sum_{t\in N_3}|a_{it}|\frac{R_t(A)}{|a_{tt}|}\right] \quad (7)$$

and at least one strict inequality holds in Equation (7), then A is a non-singular H-matrix.

Proof:

According to Equation (7), there is

$$|a_{ii}|\frac{\Lambda_i(A)-|a_{ii}|}{\Lambda_i(A)} \geq M\sum_{t\in N_1}|a_{it}|+\sum_{\substack{t\in N_2\\ t\neq i}}|a_{it}|\frac{\Lambda_t(A)-|a_{tt}|}{\Lambda_t(A)}+\sum_{t\in N_3}|a_{it}|\frac{R_t(A)}{|a_{tt}|}, \quad (8)$$

Let us construct a diagonally dominant positive-definite matrix $X = diag(x_1, x_2, \ldots x_n)$, where

$$x_i=\begin{cases} M \ , & i\in N_1 \\ \dfrac{\Lambda_i(A)-|a_{ii}|}{\Lambda_i(A)}, & i\in N_2 \\ \dfrac{R_i(A)}{|a_{ii}|}, & i\in N_3 \end{cases}$$

Let $B = AX = (b_{ij})_{nxn}$. Whereas $M \geq \frac{\Lambda_t(A)-|a_{tt}|}{\Lambda_t(A)}(t\in N_2)$, $M\geq \frac{R_t(A)}{|a_{tt}|}(t\in N_3)$, for $\forall i\in N_1$

$$\Lambda_i(B)=M\sum_{\substack{t\in N_1\\ t\neq i}}|a_{it}|+\sum_{t\in N_2}|a_{it}|\frac{\Lambda_t(A)-|a_{tt}|}{\Lambda_t(A)}+\sum_{t\in N_3}|a_{it}|\frac{R_t(A)}{|a_{tt}|}\leq M\sum_{\substack{t\in N_1\\ t\neq i}}|a_{it}|+M\sum_{t\in N_2}|a_{it}|+M\sum_{t\in N_3}|a_{it}|$$

$$=M\Lambda_i(A)=M|a_{ii}|=|b_{ii}|$$

According to Equation (8), for $\forall i\in N_2$,

$$\Lambda_i(B)=M\sum_{t\in N_1}|a_{it}|+\sum_{\substack{t\in N_2\\ t\neq i}}|a_{it}|\frac{\Lambda_t(A)-|a_{tt}|}{\Lambda_t(A)}+\sum_{t\in N_2}|a_{it}|\frac{R_t(A)}{|a_{tt}|}\leq |a_{ii}|\frac{\Lambda_i(A)-|a_{ii}|}{\Lambda_i(A)}=|b_{ii}|$$

Since at least one strict inequality holds in Equation (7), there exists $i_0\in N_2$ such that $|b_{i_0i_0}|>\Lambda_{i_0}(B)$. Whereas $K\geq \frac{R_i(A)}{|a_{ii}|}(i\in N_3)$, for $\forall i\in N_3$,

$$\Lambda_i(B) = M\sum_{t\in N_1}|a_{it}| + \sum_{t\in N_2}|a_{it}|\frac{\Lambda_t(A)-|a_{tt}|}{\Lambda_t(A)} + \sum_{\substack{t\in N_3\\ t\neq i}}|a_{it}|\frac{R_t(A)}{|a_{tt}|}$$

$$\leq M\sum_{t\in N_1}|a_{it}| + \sum_{t\in N_2}|a_{it}|\frac{\Lambda_t(A)-|a_{tt}|}{\Lambda_t(A)} + K\sum_{\substack{t\in N_3\\ t\neq i}}|a_{it}| = R_i(A) = |a_{ii}|\frac{R_i(A)}{|a_{ii}|} = |b_{ii}|$$

The irreducibility of A reveals that $B = AX$ is irreducible. According to Lemma 1, B is a non-singular H-matrix. Therefore, it is concluded that A is a non-singular H-matrix.

Theorem 3

Assuming that $A = (a_{ij}) \in M_n(c)$ and $|a_{ii}| \neq \sum_{\substack{t\in N_1\\ t\neq i}}|a_{it}|\,(\forall i \in N_1)$. If, for $\forall i \in N_2$, there is

$$|a_{ii}| \geq \frac{\Lambda_i(A)}{\Lambda_i(A)-|a_{ii}|}\left[M\sum_{t\in N_1}|a_{it}| + \sum_{\substack{t\in N_2\\ t\neq i}}|a_{it}|\frac{\Lambda_t(A)-|a_{tt}|}{\Lambda_t(A)} + \sum_{t\in N_2}|a_{it}|\frac{R_t(A)}{|a_{tt}|}\right]$$

and for the i that is true for Equation (8), there exists a chain of nonzero elements $a_{ij_1}a_{j_1j_2}\ldots a_{j_{k-1}j_k} \neq 0$ such that

$$|a_{kk}| > \frac{\Lambda_k(A)}{\Lambda_k(A)-|a_{kk}|}\left[M\sum_{t\in N_1}|a_{it}| + \sum_{\substack{t\in N_2\\ t\neq i}}|a_{it}|\frac{\Lambda_t(A)-|a_{tt}|}{\Lambda_t(A)} + \sum|a_{it}|\frac{\Lambda_t(A)}{|a_{tt}|}\right]$$

then, A is a non-singular H-matrix.

Proof:

Let us construct a diagonally dominant positive-definite matrix X similar to the proof of Theorem 2. It is easy to prove that, for $\forall i \in N$, there is $|b_{ii}| \geq \Lambda_i(B)$ and $J(B) = \{i \in N \,\|b_{ii}| > \Lambda_i(B)\} \neq \varnothing$, and, for vertex i satisfying $|b_{ii}| = \Lambda_i(B)$, there exists a chain of nonzero elements $b_{ir_1}a_{r_1r_2}\ldots a_{r_pj} \neq 0$ such that $i \in J(A)$. That is to say, B is a diagonally dominant matrix with a chain of nonzero elements. According to Lemma 2, B is a non-singular H-matrix. Therefore, it is concluded that A is a non-singular H-matrix.

Note: It is clear that Theorems 2 and 3 here contain Theorems 2 and 3 in Reference [1].

3. Numerical Example

The following matrix is taken as an example:

$$A=\begin{pmatrix} 3 & 1 & 1 & 0 & 1 \\ 2 & 4 & 0 & 1 & 1 \\ 0 & 1 & 8 & 1 & 10 \\ 1 & 1 & 0 & 9 & 1 \\ 1 & 0 & 1 & 1 & 7 \end{pmatrix}$$

In this case, $N_1=\{1,2\}$, $N_2=\{3\}$, $N_3=\{4,5\}$

Whereas $|a_{11}|=3\neq\sum_{\substack{t\in N_1\\ t\neq 1}}|a_{1t}|=1$, $|a_{22}|=4\neq\sum_{\substack{t\in N_1\\ t\neq 2}}|a_{2t}|=2$

$$\frac{\Lambda_3(A)}{\Lambda_3(A)-|a_{33}|}\left[M\left(|a_{31}|+|a_{32}|\right)+|a_{34}|\frac{R_4(A)}{|a_{44}|}+|a_{35}|\frac{R_5(A)}{|a_{55}|}\right]=3\left(\frac{3}{7}+\frac{\frac{62}{63}}{9}+\frac{\frac{56}{63}}{7}\times 10\right)=5\frac{240}{567}<8=|a_{33}|$$

Hence, A satisfies the condition in Theorem 1, indicating that it is a non-singular H-matrix. Whereas

$$\frac{\Lambda_3(A)}{\Lambda_3(A)-|a_{33}|}\left(|a_{31}|+|a_{32}|+|a_{34}|\frac{P_4(A)}{|a_{44}|}+|a_{35}|\frac{P_5(A)}{|a_{55}|}\right)=3\left(1+\frac{\frac{7}{3}}{9}+10\times\frac{\frac{7}{3}}{7}\right)=13\frac{7}{9}>8=|a_{33}|$$

Thus, it is impossible to determine A with the theorems in Reference [1].

4. Annex

The $M,k,R_4(A),R_5(A),P_4(A),P_5(A),r$ in the example are calculated as follows:

Whereas

$$\frac{\Lambda_3(A)-|a_{33}|}{\Lambda_3(A)}=\frac{1}{3},\frac{\Lambda_4(A)}{|a_{44}|}=\frac{1}{3},\frac{\Lambda_5(A)}{|a_{55}|}=\frac{3}{7}$$

Thus

$$M = \max_{\substack{i \in N_2 \\ j \in N_3}} \left\{ \frac{\Lambda_i(A) - |a_{ii}|}{\Lambda_i(A)}, \frac{\Lambda_j(A)}{|a_{jj}|} \right\} = \frac{3}{7}$$

Whereas

$$\frac{M\left(|a_{41}| + |a_{42}|\right) + |a_{43}| \dfrac{\Lambda_3(A) - |a_{33}|}{\Lambda_3(A)}}{|a_{44}| - |a_{45}|} = \frac{\frac{3}{7} \times 2}{9 - 1} = \frac{3}{28}$$

$$\frac{M\left(|a_{51}| + |a_{52}|\right) + |a_{53}| \dfrac{\Lambda_3(A) - |a_{33}|}{\Lambda_3(A)}}{|a_{55}| - |a_{45}|} = \frac{\frac{3}{7} + \frac{1}{3}}{6} = \frac{8}{63}$$

Thus

$$K = \max_{i \in N_3} \left(\frac{M \sum_{i \in N_1} |a_{it}| + \sum_{i \in N_2} |a_{it}| \dfrac{\Lambda_t(A) - |a_{tt}|}{\Lambda_i(A)}}{|a_{ii}| - \sum_{\substack{i \in N_3 \\ t \neq i}} |a_{it}|} \right) = \frac{8}{63}$$

$$R_4 = M\left(|a_{41}| + |a_{42}|\right) + K|a_{45}| = \frac{3}{7}(1+1) + \frac{8}{63} = \frac{62}{63}$$

$$R_5 = M\left(|a_{51}| + |a_{52}|\right) + |a_{53}| \frac{\Lambda_3(A) - |a_{33}|}{\Lambda_3(A)} + K|a_{54}| = \frac{3}{7} + 1 \times \frac{1}{3} + \frac{8}{63} \times 1 = \frac{56}{63}$$

Whereas

$$\frac{|a_{41}| + |a_{42}| + |a_{43}|}{|a_{44}| - |a_{45}|} = \frac{2}{8} = \frac{1}{4}$$

$$\frac{|a_{51}| + |a_{52}| + |a_{53}|}{|a_{55}| - |a_{54}|} = \frac{2}{6} = \frac{1}{3}$$

Thus

$$r = \max_{i \in N_3} \left(\frac{\sum_{i \in N_1} |a_{it}| + \sum_{i \in N_2} |a_{it}|}{|a_{ii}| - \sum_{\substack{i \in N_3 \\ t \neq i}} |a_{it}|} \right) = \frac{1}{3}$$

$$P_4(A) = |a_{41}| + |a_{42}| + |a_{43}| + r|a_{45}| = 2 + \frac{1}{3} = \frac{7}{3}$$

$$P_5(A) = |a_{51}| + |a_{52}| + |a_{53}| + r|a_{54}| = 2 + \frac{1}{3} = \frac{7}{3}$$

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