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Optimal Pricing and Admission Control of Markovian Queueing System with Negative Customers

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Abstract

This paper analyses the optimal dynamic pricing and admission control policies to maximize the average benefit in a Markovian queue with negative customers. The negative customers, as a type of job cancellation signals, are frequently employed to solve the congestion problem in the production system. In our model, the manager proposes a price for positive customers, and decide whether or not to accept the arriving negative customers in any decision epoch. Treating the problem as a Markov decision process, the author derived the monotonicity of the optimal pricing policy, proved the optimal admission policy as a threshold policy, and verified the monotonicity of the threshold policy in system parameters. Finally, some numerical experiments were presented to depict the effect of system parameters on the optimal policy and average benefit.

Key words

Queueing system, Dynamic pricing, Admission control, Markov decision process, Negative customers.

1. Introduction

Recently, there has been a growing interest in Markovian queueing systems with negative customers. Unlike ordinary customers, the negative customers require no service and reduce a queue of ordinary customers a nonempty queueing system [1]. Over the years, queueing models with negative customers into have been extensively applied in performance optimization of production inventory systems, service organizations and computer systems. The implementation has aroused wide-ranging theoretical interests and given birth to diverse practical applications. In signal systems, negative customers are represented as inhibition signals, i.e., the instructions to cancel requests for resources [2,3]. In database systems, negative customers act as instructions to halt the operations made impossible by data locking [4-6]. In neural networks, negative and positive customers serve as inhibitory and excitatory signals, respectively. In inventory systems, negative customers stand for signals to dispose items in the serviceable inventory [7,8].

To improve the management of queueing systems in different industries, much research has been done on the dynamic pricing and admission control problems. However, rarely has any scholar explored the dynamic control of the queues with negative customers. Considering the popularity of such queues, it is meaningful to study the optimal control of the queues with negative customers. The purpose of dynamic pricing is to enhance network manager's ability to recover costs and make benefits, thus promoting capacity expansions. In optimal pricing problems, the customers are assumed to accept a highest consumption price, which is a random variable called the reservation price, and the manager is assumed to state a price at any decision epoch [9,10].

Low [11] pioneered the study of dynamic optimal pricing problems. He derived the monotonicity of the optimal prices in the queue length. Son [10] examined the optimal pricing control problem from the perspectives of deterministic service times and side-line benefit. Yoon and Lewis [12] disclosed the monotonicity of a queueing system with periodically varying parameters. Similar monotonicity results were also derived for the make-to-stock queue model in a production inventory system [13]. Cil et al. [14] explored an optimal dynamic pricing problem for a two-class queueing system, concluding that the optimal pricing control depends on the queue length vector. Feinberg et al. [15] studied the optimal pricing of a GI/M/k/N queue involving different types of customers and holding costs.

Being a provisioning strategy to limit the number of customers in a system, admission control is essential to packet-switched networks, as it is capable of relieving the traffic congestion. Heyman [16] was the first to study the optimal admission control problems. The early papers on admission control of queueing systems were summarized by Stidham [17]. Yoon and Lewis [12] opened the new research field of admission control in periodic nonstationary queueing systems. Son [10] discussed the optimal admission control of a service company with two classes of customers. Wu et al. [18] investigated the multiple product admission control in semiconductor production systems under the constraint of process queue time (PQT).

The structural properties of optimal pricing and admission control were widely discussed by Koole [19] and Lin et al. [20]. As far as we know, however, no report has been released on the pricing policy and admission control of queues with negative customers prior to our research. To make up for the gap, this paper probes into the structure of optimal pricing and admission control policies in a queueing system with negative customers. The goal is to find the optimal policy that yields the maximum average benefit over an infinite horizon. To this end, the system manager must weigh the penalty and holding cost against reward. Furthermore, this research is motivated by the vision that the wide applications of negative customers may offer a mechanism to curb the excessive congestion of production inventory systems. Specifically, the pricing control of positive customers were considered as the balk behaviour of customer demand, the negative customers were regarded as the disposal of items or the transition to the secondary market, and the manager decided whether or not to accept the negative customers, seeking to reduce the excess items in the inventory. The research findings help to improve the management of inventory systems, and enable the manager to achieve the maximum average benefit via the optimal control policy.

The main contributions of this research are as follows. First, to the best of our knowledge, this research is the first to investigate the optimal pricing and admission control policies in a queueing system with negative customers, which fills a gap in the research into the control of queues with negative customers. Second, the author derived the structure of optimal policy and the monotonicity properties of the optimal pricing and admission threshold. Third, the results obtained in this research were verified by the numerical results acquired by the Howard’s iteration procedure [21].

The remainder of this paper is organized as follows. Section 2 formulates the model based on the controllable Markov decision process and derives the optimality equation; Section 3 discusses the structural properties of the optimal policy in the model; Section 4 examines the effect of system parameters on the optimal policy and average cost based on several numerical examples; Section 5 wraps up the research with further discussions and conclusions.

2. Model Description

This research focuses on a single-server first-come, first-served (FCFS) queueing system with negative customers. In the system, the arrivals of positive customers and negative customers are two independent Poisson processes with the rate of λ^+ and λ^-, respectively. the service time of each positive customer is exponentially distributed with rate parameter μ

For every fixed proposed price r ($r \in [r_{min}, r_{max}]$), whenever a positive customer arrives, he/she either enters the system if his/her reservation price Q exceeds the proposed price or leaves the system without receiving any reward. It is assumed that Q is a random variable for which the distribution function and density function are denoted by W(.) and w(.), respectively, and that $\overline{W}(r)=1-W(r)$. The system incurs a holding cost h per positive customer per unit of time. Whenever a negative customer arrives, he/she is either accepted or rejected by the manager. If the negative customer is accepted, a positive customer will be killed immediately, and the system will incur a penalty cost c for the killed customer.

The set of decision epochs consists of the set of all arrivals, service completions, and dummy transitions due to normalization. In any decision epoch, the manager has to choose a proposed price r from the set A= $[r_{min}, r_{max}]$ and decide whether or not to accept the negative customer. If the number of customers in the system at time t is denoted as X(t), then the system evolves as a continuous-time Markov chain $\{X(t), t \geq 0\}$ under any fixed control policy π. It is clear that the system state space is E= $\{0, 1, 2, \ldots\}$. Due to the Markovian property, the optimal policy depends only on the current state.

The manager is responsible for finding the optimal policy to maximize the long-term average benefit based on the number of customers in the system. Treating the problem as a Markov decision process, the author built a discrete-time equivalent of the original queueing system through normalization. Without loss of generality, it is assumed that that $\lambda^{+}+\lambda^{-}+\mu=1$. Thus, the total expected benefit can be obtained as:

$$V_T^{\pi}(x) = E_x^{\pi}[\int_0^T hX(t)dt + \int_0^T r(t)dM(t) + \int_0^T cdN(t)],$$

where π is the policy; x is the initial state; n is the number of horizons; E^{π}_{x} is the expectation on the probability measure determined by the policy and the initial state; M(t) and N(t) are the number of positive customer and negative customers who have entered the system at time t, respectively; r(t) is the proposed price at time t. The expectation must exist because the rewards are bounded and non-negative.

Under the assumption that $\lambda^{+}\overline{W}(r_{max}) \leq \mu$, the resulting system is a stable queueing system of finite average queue length and finite average benefit. Assuming that the process $\{X(t), t \geq 0\}$ with state space E is an irreducible, positive recurrent Markov process at each fixed stationary policy π, the long-term average benefit of the ergodic Markov process under the policy π can be written as below in light of Tijms [22].

$$g(\pi) = \lim_{T\to\infty} V_T^{\pi}(x)/T = \sum_{i\in E}[r(i,\pi(i))]p_i(\pi),$$

where pi(π) is a stationary probability of the system under policy π; r (x, a) is the expected benefit of the system in state x and action a. Let ∏ be the set of all admission policies. The goal is to find the optimal policy π* that maximizes the long-term average benefit:

$$g = g(\pi^*) = \max_{\pi\in\Pi} g(\pi).$$

To find such a policy, a real-valued function v(x) is defined in the state space. The relative value function is regarded as the asymptotic difference in total costs if the process starts in state x instead of some reference states. According to Puterman [21], the optimal policy π* and the optimal average benefit g are the solutions of the optimality equation below:

$$Tv(x) = v(x) + g, \tag{1}$$

where T is the dynamic programming operator acting on v. The relevant operators are defined as:

$$T_P v(x) = \max_{r\in A}\{\overline{W}(r)[v(x+1)+r] + W(r)v(x)\},$$
$$T_A v(x) = max\{v(x-1) - c, v(x)\}.$$

The first operator T_P v(x) simulates the admission control of the arriving positive customers based on optimal pricing; the second operator T_A v(x) simulates the admission control of arriving negative customers based on value variation.

The first step to examine the properties of the optimal policy is to investigate the properties of the relative value function. The key lies in the analysis of the operators T_P and T_A. The properties of the relative value function v(x) are defined as follows:

Decreasing: $v(x) \geq v(x+1)$,

Concavity: $2v(x) - v(x+1) - v(x-1) \geq 0$,

Convexity: $2v(x) - v(x+1) - v(x-1) \leq 0$.

Referring to Cil et al. [9], it is assumed that a function v(x) has a certain property ϕ preserved by operator T if the property also belongs to Tv(x). Through the deduction on n in v(x), it is possible to acquire the properties of the operators T_P and T_A in our model that preserve the desired properties (decreasing, concavity, convexity) for the function v(x).

3. Structure of the Optimal Control Policy

This section attempts to derive the optimal policy. The properties of the optimal policy helps to reduce the solution search space, and ease the computing load in the search of the optimal policy.

The optimality equation (2.1) should be solved before exploring the optimal policy. Whereas it is hard to solve the equation analytically, the $v_{n+1}=Tv_n$ is recursively defined for a random v_0 based on the system state transition rate, the stochastic dynamic programming, and the induction method. It is known that the actions converge to the optimal policy as $n\rightarrow\infty$. The existence and convergence of the solutions and optimal policy have been detailed by Aviv and Federgruen [23] and Sennott [24]. The backward recursion equation is expressed as:

$$v_{n+1}(x) = \lambda^{+} T_P v_n(x) + \lambda^{-} T_A v_n(x) + \mu v_n(x-1) - xh. \qquad (2)$$

The main properties of the operators in the system can be summarized by the following lemma (the proof is given in the Appendix).

Lemma 3.1 For the relative value function v(x) in the model, we have:

(1) The operator T_P preserves the properties: Decreasing, Concavity, Convexity,

(2) The operator T_A preserves the properties: Decreasing, Concavity, Convexity.

According to the backward recursion equation (3.1), the following properties of the relative value function v(x) can be obtained based on the above properties of the operators and the induction method:

$$v(x) \geq v(x+1),\ 2v(x) - v(x+1) - v(x-1) \geq 0.$$

On the basis of the structure properties of the relative value function v(x), the structure of the optimal pricing policy is expressed in the following theorem. Please refer to Çl, E.B [9] for the proof of the theorem.

Theorem 3.1. The optimal pricing control policy has the following properties:

(1) If $\overline{W}(r)/(r\,W(r))$ is strictly decreasing in $r\in[r_{min}, r_{max}]$, then the optimal pricing is unique;

(2) If the optimal pricing is non-decreasing in $x \in E$, then $r^*(x) \leq r^*(x+1)$ for $x \in E$.

Next, the structure of the optimal admission policy was discussed and some conditions were given to ensure the simplicity of the policy in the model. As mentioned above, the properties of the optimal policy helps to reduce the solution search space, and ease the computing load in the search of the optimal policy. Specifically, the structure of the optimal policy was converted as the properties of the optimal value function and the optimality equation. Following the optimality equation, the operator T_A can be rewritten as:

$$T_A v(x) = max\{v(x-1) - c, v(x)\} = max\{H(x-1), 0\} + v(x),$$

where $H(x-1) = v(x-1) - v(x) - c$.

From the above equations, it can be seen that the properties of $H(x)$ should be examined before deriving the structure of the optimal admission policy. For this purpose, the author presented the following lemma (the proof is given in the Appendix).

Lemma 3.2. For the admission control problem in the present model, we have:

(1) The function $H(x)$ is increasing for all $x \in E$.

(2) If the condition $h/\mu \geq c$ holds, then $H(0) \geq 0$.

The admission control problem was analysed in two aspects. First, the acceptance of an arriving negative customer will incur a penalty cost c and the removal of a positive customer. Second, the rejection of an arriving negative customer will incur a holding cost to the positive customer. Hence, the system manager must weigh the penalty and holding cost against reward. This means the decision depends on the number of customers in the system and the parameters h and c .

Theorem 3.2. The optimal admission policy is a threshold policy, that is, the negative customer should be rejected if $x < N^*$ and be accepted if otherwise; $N^* = \min\{x: H(x) \geq 0\}$; $N^* = 0$ if the condition $h/\mu \geq c$ holds.

Proof Since the function $H(x)$ is increasing for all $x \in E$ (Lemma 3.2 (1)), there must exist an optimal threshold policy for the admission control problem. Concretely, there exists an N^* such that $H(x) \leq 0$ for all states $x \leq N^*$ and $H(x) \geq 0$ for all states. Moreover, it states that the negative customer should be rejected if $x \leq N^*$ and be accepted if otherwise; By the definition of the admission operator T_A, the threshold parameter is $N^* = \min\{x: H(x) \geq 0\}$; $N^* = 0$ if the condition $h/\mu \geq c$ holds. From Lemma 3.2 (2), it is obtained that $H(x) \geq 0$ for all states $x \in E$. Hence, the optimal admission policy is a pure reception policy, i.e., the negative customer should be accepted for all $x \in E$.

Intuitively, it is learned that the minimum holding cost is h/μ for a positive customer. If the condition $h/\mu \geq c$ holds, the minimum holding cost for a positive customer must surpass the penalty cost. To remove the positive customer from the system, the manager has to accept the negative customer. However, if the condition $H(0) \leq 0$ holds, the manager should decide whether or not to accept the negative customer based on the number of positive customers in the system. Therefore, there exists a threshold $N^*>0$ such that the negative customers should be rejected for the states $x \geq N^*$.

Next, the monotonicity properties of the two thresholds m and n were discussed with respect to various system parameters. Referring to the method in Benjaafar et al. [25] and C¸il et al. [14], the author compared the optimal value functions of two systems which are identical except for the value of one parameter, denoted as q. The optimal admission thresholds and optimal value function corresponding to q are represented by N_q and $v_q(x)$, respectively, where $q \in \{\lambda^+, \lambda^-, c, h\}$.

In order to derive the monotonicity properties of the two thresholds, the properties of the optimal value function $v_q(x)$ in the two systems were examined in light of Koole [19]. To make the two systems comparable, the normalization rate, depending on $\{\lambda^+, \lambda^-, \mu\}$, must be constant. The time was rescaled by a normalization rate τ sufficiently greater than the $\lambda + \mu + \xi$ so that q and $q + \varepsilon$ share the same normalization rate. To maintain a constant normalization rate, the fictitious event in the two system is τ-q and τ-q-ε, respectively. For instance, if $q=\mu$, the optimality recursion equations of the system with parameter μ and the system with $\mu + \varepsilon$ are respectively expressed as:

$$v_\mu(x) = \frac{1}{\tau}[\lambda^+ T_P v_\mu(x) + \lambda^- T_A v_\mu(x) + \mu v_\mu(x-1) + (\tau - \lambda^+ - \lambda^- - \mu)v_\mu(x) - xh]. \quad (3)$$

$$v_{\mu+\epsilon}(x) = \frac{1}{\tau}[\lambda^+ T_P v_{\mu+\epsilon}(x) + \lambda^- T_A v_{\mu+\epsilon}(x) + \mu v_{\mu+\epsilon}(x-1) + (\tau - \lambda^+ - \lambda^- - \mu - \epsilon)v_{\mu+\epsilon}(x) - xh]. \quad (4)$$

where T_P and T_A are defined in the previous section. By this method, the following lemma is arrived at (the proof is given in the Appendix).

Lemma 3.3. For the optimal value function $v_q(x)$ of the two systems with different parameters q, we have:

(1) $\forall q \in \{\lambda^+, h\}$, $\Delta v_{q+\epsilon}(x) \geq \Delta v_q(x)$,

(2) $\forall q \in \{\lambda^-, c\}$, $\Delta v_{q+\epsilon}(x) \leq \Delta v_q(x)$,

where $\Delta v_q(x)=v_q(x-1)-v_q(x)$ and $\forall \varepsilon \geq 0$.

Based on the above properties of optimal value function, the structure of the optimal policy was obtained by analysing the effect of various system parameters on the thresholds. The results are summarized in the following theorem.

Theorem 3.3. In the system control problems, the optimal admission threshold N* is non-decreasing in λ^-, c and λ^+□h.

Proof In Theorem 3.2, the admission threshold is defined as N*=min {x: H(x)≥0}. According to the comparison above and Lemma 3.3 (1), it is known that $\Delta v_{\lambda^+ + \varepsilon}(x) \geq \Delta v_{\lambda^+}(x)$ and $\Delta v_{h+\varepsilon}(x) \geq \Delta v_h(x)$. Hence, the admission threshold N* is non-increasing in λ^+, h. According to Lemma 3.3 (2), $\Delta v_{\lambda^- + \varepsilon}(x) \geq \Delta v_{\lambda^-}(x)$ and $\Delta v_{c+\varepsilon}(x) \geq \Delta v_c(x)$, indicating that the admission threshold N* is non-decreasing in λ^-, c.

4. Numerical Examples

Several numerical examples were developed similar to those in [9]. It is assumed that μ=1 and the proposed price is uniformly distributed in the interval [2, 14]. Whereas Howard's policy iteration algorithm is an effective numerical calculation tool for the Markov decision problem, the algorithm was modified [21] to handle the numerical examples. The examples were designed to reflect the effect of system state transition on the optimal pricing r*(x), verify the structure of the optimal admission threshold and average benefit obtained in Section 3, and demonstrate the response of the optimal policy and average benefit to the system parameters. The observations are presented in the table and figures below.

Tab.1. Optimal Pricing vs. x for $\lambda+=0.8$, $\lambda-=0.3$, $h=1$, $c=3$

X	$r*(x)$	x	$r*(x)$	x	$r*(x)$	x	$r*(x)$	x	$r*(x)$
0	1.00	6	1.28	12	2.93	18	5.25	24	6.00
1	1.00	7	1.50	13	3.28	19	5.60	25	6.00
2	1.12	8	1.72	14	3.65	20	5.85	26	6.00
3	1.18	9	2.05	15	4.02	21	6.00	27	6.00
4	1.20	10	2.35	16	4.32	22	6.00	28	6.00
5	1.24	11	2.62	17	4.85	23	6.00	29	6.00

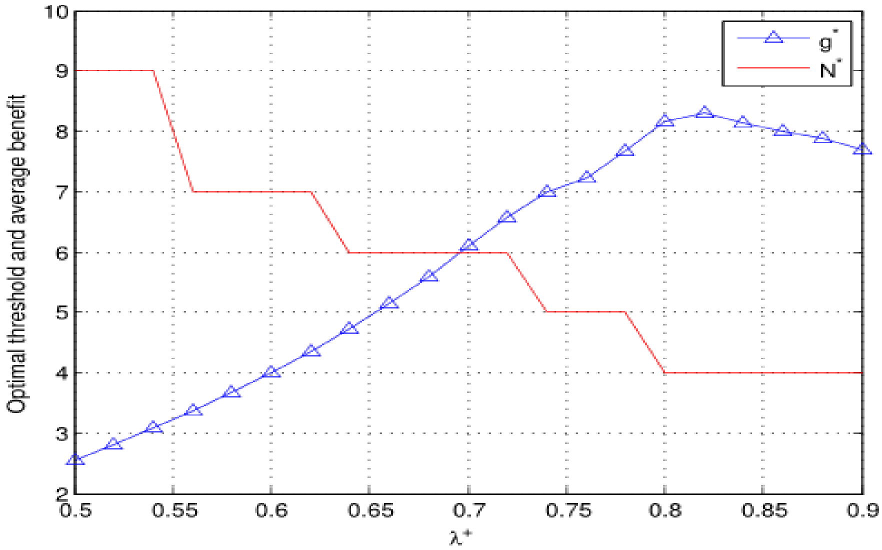

Fig.1. Optimal Threshold and Average Benefit vs. λ^+ for λ^-=0.3, h-=1, c=3

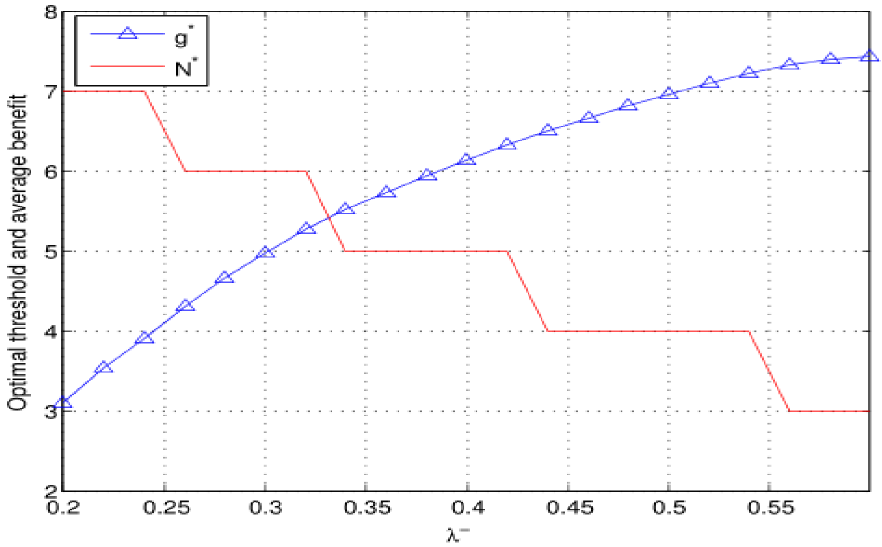

Fig.2. Optimal Threshold and Average Benefit vs. λ^- for λ^+=0.6, h-=1, c=3

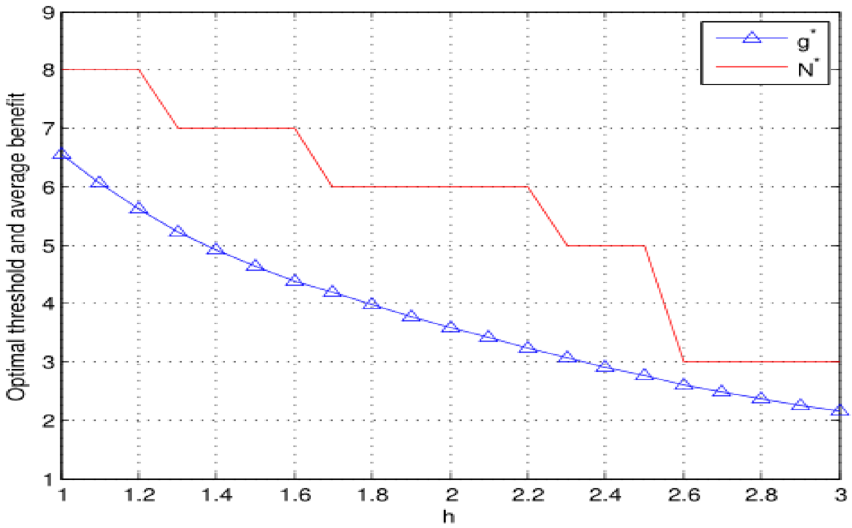

Fig.3. Optimal Threshold and Average Benefit vs. h for λ^+=0.7, λ^- =0.5, c=3

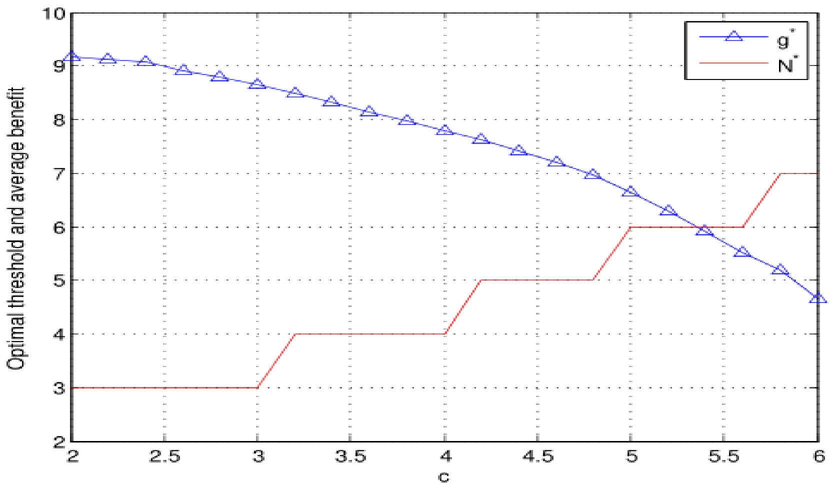

Fig.4. Optimal Threshold and Average Benefit vs. c for λ^+=0.8, λ^- =0.3, h=1

Table 1 depicts the relationship between the optimal pricing and system state. As shown in the table, the optimal pricing r*(x) increases with the number of customers in the system x. As the number grows within certain ranges, the optimal pricing will reach the maximum pricing r*(x)=6 and remain the same. The phenomena are consistent with the reality and easy to explain. For example, when the number of customers becomes sufficiently large in the system, the holding cost will grow, forcing the manager to propose the maximum pricing and reject the arriving customers.

Figures 1~4 present the numerical results on the response of the optimal policy and average benefit to the system parameters λ^+, λ^-, h and c. As shown in Figure 1, the optimal threshold decreases with the increase of λ^+, while the average benefit first increases and then decreases with the increase of λ^+. The increase of the average benefit is attributable to the system welfare brought by the arriving positive customers, while the decrease of the average benefit is resulted from the growing number of positive customers, and the ensuing growth in holding cost in a certain interval.

Figure 2 shows the effect of rate λ^-on the optimal threshold and average benefit. It can be seen that the optimal threshold increases with parameter λ^-, and the average benefits grows with ξ but at a slower rate. As can be seen from Figures 3 and 4, the optimal threshold decreases with the increase of h, while the average benefit falls with the increase of either h or c. Moreover, all the values of the optimal thresholds exhibit a staircase-like monotonous pattern, indicating that the optimal threshold is not affected by the minor changes of system parameters.

Conclusion

This paper digs into the optimal dynamic pricing and admission control policies to maximize the average benefit in a Markovian queue with negative customers. The negative customers, as a type of item removal signals, are frequently employed to solve the congestion problem in the production inventory system. Treating the problem as a Markov decision process, the author derived the monotonicity of the optimal pricing policy, proved the optimal admission policy as a threshold policy after analysing the properties of the value function, and discovered the monotonicity of the optimal thresholds to some system parameters through comparisons. Moreover, the Howard's iteration algorithm was adopted for the numerical experiments, which were designed to reveal the behaviours of optimal policies were studied under different values of system parameters. The proposed method is applicable to a wide range of models, including the optimal maintenance and production policies in the production system, and the optimal routing, scheduling and production policies in the management system.

Further investigation is needed to apply the results to simulate more complex systems. For example, the proposed model could be extended to study the optimal control problem in the queues with disaster, or implemented in systems of which the service time obeys the general distribution of the embedded Markov process and semi-Markov process. Furthermore, the optimal control of the model may be combined with the uncertainties to provide more accurate information to the manager. Such uncertainties include randomness and fuzziness, which are commonplace in actual product inventory systems.

Appendix

The proof of Lemma 3.1 (1)

Proof. To prove the decreasing property of operator T_P, let r* be the optimal price for the state x+1. Then we show that pricing operator T_P preserves the decreasing property of v(x)in x. From the definition of the pricing operator T_P, we get

$$
\begin{aligned}
\max_{r\in A}\{\overline{W}(r)[v(x+1)+r]+W(r)v(x)\} &\geq \overline{W}(r^*)[v(x+1)+r^*]+W(r^*)v(x) \\
&\geq \overline{W}(r^*)[v(x+2)+r^*]+W(r^*)v(x+1) \\
&= \max_{r\in A}\{\overline{W}(r)[v(x+2)+r]+W(r)v(x+1)\}.
\end{aligned}
$$

The first inequality follows by taking a potentially suboptimal action in state x and the second inequality is based on the decreasing property of v(x) in x. The equality follows by the definition of r^*. Hence, we have $T_P v(x) \geq T_P v(x+1)$.

To prove the concavity of operator T_P, let r_1, r_2 and r_3 be the optimal prices for the states x, x+1 and x+2. Then we show that pricing operator T_P preserves the concavity of v(x) in x. From the definition of the pricing operator T_P, we get

$$
\begin{aligned}
&T_P v(x+1) - T_P v(x+2) \\
&\quad = \overline{W}(r_2)[v(x+2)+r_2]+W(r_2)v(x+1)-\overline{W}(r_3)[v(x+3)+r_3]-W(r_3)v(x+2) \\
&\quad \geq \overline{W}(r_1)[v(x+2)+r_1]+W(r_1)v(x+1)-\overline{W}(r_3)[v(x+3)+r_2]-W(r_3)v(x+2) \\
&\quad = \overline{W}(r_1)r_1-\overline{W}(r_3)r_3+W(r_1)[v(x+1)-v(x+2)]+\overline{W}(r_3)[v(x+2)-v(x+3)].
\end{aligned}
$$

The first equality is based on the definition of the pricing operator and the inequality follows by taking a potentially suboptimal action in state x+1.

$$
\begin{aligned}
&T_P v(x) - T_P v(x+1) \\
&\quad = \overline{W}(r_1)[v(x+1)+r_1]+W(r_1)v(x)-\overline{W}(r_2)[v(x+2)+r_2]-W(r_2)v(x+1) \\
&\quad \leq \overline{W}(r_1)[v(x+1)+r_1]+W(r_1)v(x)-\overline{W}(r_3)[v(x+2)+r_2]-W(r_3)v(x+1) \\
&\quad = \overline{W}(r_1)r_1-\overline{W}(r_3)r_3+W(r_1)[v(x)-v(x+1)]+\overline{W}(r_3)[v(x+1)-v(x+2)].
\end{aligned}
$$

The first equality is based on the definition of the pricing operator and the inequality follows by taking a potentially suboptimal action in state x+1 and the second equality is based on arranging the terms. Because of the concavity property of v(x), we have $2v(x)-v(x+1)-v(x+2)\geq 0$ and $2v(x+1)-v(x+2)-v(x+3)\geq 0$. Hence, we get $2T_P v(x)-T_P v(x+1)-T_P v(x+2) \geq 0$.

To prove the convexity of operator T_P, let r be the optimal price for the state x. Then we show that T_P preserves the convexity of $v(x)$ in x. From the definition of T_P, we get

$$
\begin{aligned}
&2T_P v(x) - T_P v(x+1) - T_P v(x-1) \\
&\quad \leq 2[\overline{W}(r)[v(x+1)+r]+W(r)v(x)]-\overline{W}(r)[v(x+2)+r] \\
&\qquad -W(r)v(x+1)-\overline{W}(r)[v(x)+r]-W(r)v(x-1) \\
&\quad = \overline{W}(r)[2v(x+1)-v(x+2)-v(x)]+W(r)[2v(x)-v(x+1)-v(x-1)] \leq 0.
\end{aligned}
$$

The first inequality follows by taking a potentially suboptimal action in states x-1 and x+1. The equality is based on arranging the terms and the second inequality is based on the assumption.

The proof of Lemma 3.1 (2)

Proof. To prove the decreasing property of operatorT_A, we can get it from the decreasing property of v(x) and the definition of the admission operator T_A. We omit the details here. To prove the concavity of operator T_A, from the definition of operator T_A, we get

$$\begin{aligned}&2T_Av(x) - T_Av(x+1) - T_Av(x-1)\\&\quad = 2max\{v(x-1) - c, v(x)\} - max\{v(x) - c, v(x+1)\} - max\{v(x-2) - c, v(x-1)\}.\end{aligned}$$

Because of the concavity property of v(x), the above equation has four cases:

$$\begin{aligned}(a)\quad & 2T_Av(x) - T_Av(x+1) - T_Av(x-1)\\&\qquad = 2v(x) - v(x+1) - v(x-1) \geq 0;\\(b)\quad & 2T_Av(x) - T_Av(x+1) - T_Av(x-1)\\&\qquad = 2(v(x-1) - c) - (v(x) - c) - (v(x-2) - c) = 2v(x-1) - v(x) - v(x-2) \geq 0;\\(c)\quad & 2T_Av(x) - T_Av(x+1) - T_Av(x-1)\\&\qquad = 2(v(x-1) - c) - (v(x) - c) - v(x-1) = v(x-1) - v(x) - c \geq 0;\\(d)\quad & 2T_Av(x) - T_Av(x+1) - T_Av(x-1)\\&\qquad = 2v(x) - (v(x) - c) - v(x-1) = v(x) - v(x-1) + c \geq 0.\end{aligned}$$

Due to the concavity of v(x), the cases (a) and (b) hold. Since we have v(x-1)-c≥v(x) in case (c) and v(x-1)-c≥v(x) in case (d), the cases (c) and (d) hold. Therefore we get $2T_A$ v(x)- T_A v(x+1)- T_A v(x-1) ≥0, i.e., the operator T_A preserves the concavity of v(x).

The proof of Lemma 3.2

Proof. To prove Lemma 3.2 (1), from Lemma 3.1, we know that 2v(x)-v(x+1)-v(x-1)≥0, which implies that the function H(x) is increasing for all x∈E.

To prove Lemma 3.2 (2), the proof is by induction on n in $v_n(x)$. Define $v_0(x)$=-cx for all states x∈E. This function satisfies the property $v_n(0)$-$v_n(1)$-c≥0. Now, we assume $v_n(0)$-$v_n(1)$-c≥0.One has to prove that $v_{n+1}(x)$satisfies the property $v_{n+1}(0)$-$v_{n+1}(1)$-c≥0 as well. Let r_0, r_1 be the optimal prices for the state 0 and 1, respectively in the model. Based on the equation (3.1), we have:

$$v_{n+1}(0) = \lambda^+[\overline{W}(r_0)[v_n(1) + r_0] + W(r_0)v_n(0)] + \lambda^- v_n(0) + \mu v_n(0),$$

$$v_{n+1}(1) = \lambda^+[\overline{W}(r_1)[v_n(2) + r_1] + W(r_1)v_n(1)] + \lambda^-[v_n(0) - c] + \mu v_n(0) - h.$$

Rearranging the terms above, we get

$$\begin{aligned} v_{n+1}(0) - v_{n+1}(1) = {} & \lambda^+[\overline{W}(r_0)[v_n(1) + r_0] + W(r_0)v_n(0)] + \lambda^- v_n(0) + h \\ & - [\lambda^+[\overline{W}(r_1)[v_n(2) + r_1] + W(r_1)v_n(1)] + \lambda^-[v_n(0) - c]] \\ \geq {} & \lambda^+\overline{W}(r_1)[v_n(1) - v_n(2)] + \lambda^+ W(r_1)[v_n(0) - v_n(1)] + \lambda^- c + h \\ \geq {} & \lambda^+[v_n(0) - v_n(1)] + \lambda^- c + h \geq (\lambda^+ + \lambda^-)c + h \geq c. \end{aligned}$$

The first in equality follows by taking a potentially suboptimal action in state x and the second inequality based on the concavity property of v(x), i.e, $v_n(1)$-$v_n(2) \geq v_n(0)$-$v_n(1)$. The third inequality follows by the assumption $v_n(0)$-$v_n(1)$-c≥0 and the last inequality based on the conditions h/μ ≥c and $\lambda^+ + \lambda^- + \mu = 1$. Therefore, we have v(0)-v(1)-c≥0.

The proof of Lemma 3.3

Proof. From the definition of the operators T_P and T_A, we get that the first order differences for the operators can be written as follows:

$$\Delta T_P v(x) = \max_{r\in A}\{\overline{W}(r)[v(x) + r] + W(r)v(x-1)\} - \max_{r\in A}\{\overline{W}(r)[v(x+1) + r] + W(r)v(x)\}$$
$$\Delta T_A v(x) = max\{v(x-2) - c, v(x-1)\} - max\{v(x-1) - c, v(x)\}.$$

In order to prove the properties, we mainly use the fixed point theorem and the iterative induction method. As the properties have the similar structure, we just consider the case $q=\lambda^+$ and the other cases can be proved in this way. We first show that the operators T_P and T_A preserve the property $\Delta v_{\lambda^+ +\varepsilon}(x) \geq \Delta v_{\lambda^+}(x)$.i.e $\Delta T_P v_{\lambda^+ +\varepsilon}(x) \geq \Delta T_P v_{\lambda^+}(x)$, $\Delta T_A v_{\lambda^+ +\varepsilon}(x) \geq \Delta T_A v_{\lambda^+}(x)$.

The proof of the result $\Delta T_A v_{\lambda^+ +\varepsilon}(x) \geq \Delta T_A v_{\lambda^+}(x)$ can be found in [12]. Next we will give the proof of the result $\Delta T_P v_{\lambda^+ +\varepsilon}(x) \geq \Delta T_P v_{\lambda^+}(x)$. Let r_1 and r_2 be the optimal prices for the states x in the model with $\lambda^+ + \varepsilon$ and x-1 in the model with λ^+ respectively. Then we show that pricing operator T_P preserves the property $\Delta v_{\lambda^+ +\varepsilon}(x) \geq \Delta v_{\lambda^+}(x)$. From the definition of the pricing operator T_P ,we get

$$
\begin{aligned}
&\Delta T_P v_{\lambda^+ +\epsilon}(x) - \Delta T_P v_{\lambda^+}(x) \\
&\quad = \max_{r\in A}\{\overline{W}(r)[v_{\lambda^+ +\epsilon}(x)+r] + W(r)v_{\lambda^+ +\epsilon}(x-1)\} - \max_{r\in A}\{\overline{W}(r)[v_{\lambda^+ +\epsilon}(x+1)+r] + W(r)v_{\lambda^+ +\epsilon}(x)\} \\
&\qquad - \max_{r\in A}\{\overline{W}(r)[v_{\lambda^+}(x)+r] + W(r)v_{\lambda^+}(x-1)\} + \max_{r\in A}\{\overline{W}(r)[v_{\lambda^+}(x+1)+r] + W(r)v_{\lambda^+}(x)\} \\
&\quad \geq \overline{W}(r_2)[v_{\lambda^+ +\epsilon}(x)+r_2] + W(r_2)v_{\lambda^+ +\epsilon}(x-1) - \overline{W}(r_1)[v_{\lambda^+ +\epsilon}(x+1)+r_1] - W(r_1)v_{\lambda^+ +\epsilon}(x) \\
&\qquad -\overline{W}(r_2)[v_{\lambda^+}(x)+r_2] - W(r_2)v_{\lambda^+}(x-1) + \overline{W}(r_1)[v_{\lambda^+}(x+1)+r_1] + W(r_1)v_{\lambda^+}(x) \\
&\quad = \overline{W}(r_2)r_2 - \overline{W}(r_1)r_1 + W(r_2)[v_{\lambda^+ +\epsilon}(x) - v_{\lambda^+ +\epsilon}(x)] + \overline{W}(r_1)[v_{\lambda^+ +\epsilon}(x) - v_{\lambda^+ +\epsilon}(x)] \\
&\qquad -\overline{W}(r_2)r_2 + \overline{W}(r_1)r_1 - W(r_2)[v_{\lambda^+ +\epsilon}(x) - v_{\lambda^+ +\epsilon}(x)] - \overline{W}(r_1)[v_{\lambda^+ +\epsilon}(x) - v_{\lambda^+ +\epsilon}(x)] \\
&\quad = W(r_2)[\Delta v_{\lambda^+ +\epsilon}(x) - \Delta v_{\lambda^+}(x)] + \overline{W}(r_1)[\Delta v_{\lambda^+ +\epsilon}(x) - \Delta v_{\lambda^+}(x)] \geq 0.
\end{aligned}
$$

While the coefficient of the operators T_P, T_A and the uniformization rate are dependent on the parameter q when q ∈ $\{\lambda^+, \lambda^-\}$.We need to show the following property$\Delta T_P v_{\lambda}^+ (x)-\Delta v_{\lambda}^+(x) \geq 0$.. Let r be the optimal price for the state x. From the definition of the operator T_P and the concavity property $\Delta v(x) \geq \Delta v(x-1)$, we have

$$
\begin{aligned}
\Delta T_P v(x) - \Delta v(x) &\geq \overline{W}(r)[v(x) - v(x+1)] + W(r)[v(x-1) - v(x)] - \Delta v(x) \\
&= \overline{W}(r)[2v(x) - v(x+1) - v(x-1)] \geq 0.
\end{aligned}
$$

Based on these properties above, we have the following inequality:

$$
\begin{aligned}
\Delta v_{\lambda^+ +\epsilon}(x) = \lambda^+ &\Delta T_P v_{\xi+\epsilon}(x) \quad (\geq \lambda^+ \Delta T_P v_\xi(x)) \\
&+ \lambda^- \Delta T_A v_{\xi+\epsilon}(x) \quad (\geq \lambda^- \Delta T_A v_\xi(x)) \\
&+ \mu \Delta v_{\xi+\epsilon}(x) \quad (\geq \mu \Delta_1 v_\xi(x)) \\
&+ (\tau - \lambda^+ - \lambda^- - \mu)\Delta v_{\xi+\epsilon}(x) \quad (\geq (\tau - \lambda^+ - \lambda^- - \mu)\Delta v_\xi(x)) \\
&+ \epsilon[\Delta T_P v_{\lambda^+ +\epsilon}(x) - \Delta v_{\lambda^+ +\epsilon}(x)] \quad (\geq 0).
\end{aligned}
$$

Therefore, we have $\Delta v_{\lambda^+ +\varepsilon}(x) \leq \Delta v_{\lambda}^+(x)$. Meanwhile, we can get the result $\Delta v_{h+\varepsilon}(x) \leq \Delta v_h(x)$, $\Delta v_{\lambda^- +\varepsilon}(x) \leq \Delta v_{\lambda}^-(x)$. and $\Delta v_{c+\varepsilon}(x) \leq \Delta v_c(x)$ in the same way.

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