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# Middle Income Positioning and Population Measurement Based on The Lorenz Curves

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## Abstract

In this article, a new Lorenz curve model is proposed, and the model is fitted with the given data. Thus the estimates of the parameters and the value of mean squared error(MSE), maximal absolute error(MAS), and mean absolute error(MAE) are got, It can be concluded that the proposed model is better than the classical models. Then an income space method is improved to make up the defects of the classical ones. Finally, the new Lorenz curve model and the improved income space method are applied to analyze the data from Problem E in the National Graduate Mathematical Contest in Modeling in 2013, and some favorable results are got.

## Key words

Lorenz curve, Middle income, Income distribution, *Gini* coefficient

## 1. Introduction

Over the years, the fairness of income distribution has been a widely-concerned topic, among which the proportion of middle-income population is an important index to reflect the income-distribution structure. As is known, the Lorenz curve is an effective method to measure the inequality of income distribution. The early researchers on the Lorenz models mostly focus on the accuracy of the model [1-6]. Recently, Wang et al. has developed satisfactory models for the Lorenz curves [7-9]. In this paper, based on the relative literature, an attractive new Lorenz curve model is proposed, and the proposed Lorenz model performs well under the grouped income data.

## 2. The new proposed Lorenz curve model

Generally speaking, the income distribution of a country can be represented by the statistical distribution. Figure 1 is an income distribution density function, x>0 is income (only considering positive income), $x_0$ is the mode, m is the median number, μ is the average income. *F(x)* denotes the corresponding distribution function, then *P=F(x)* is the proportion of people earning less than or equal to *x*. The ratio of *F(x)* to the total revenue is *L(p)*, thus:

$$L(p)=\frac{1}{\mu}\int_0^x tf(t)\mathrm{d}t\,,\ \ p=F(x)$$

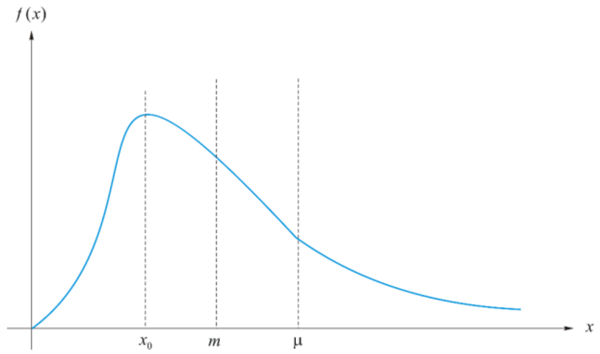


Fig.1. Income distribution density function

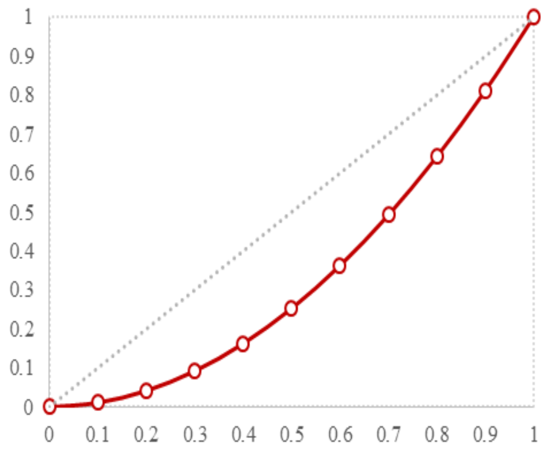


Fig.2. Lorenz curve

*L(p)* is the Lorenz curve of income distribution. In figure 2, the red curve is a Lorenz curve of income distribution,where the horizontal axis shows the population proportion, the vertical axis shows the income proportion. The relationship of *L(p)* and *f(x)* is:

$$L'(p) = \frac{x}{\mu}$$

$$f(x) = \frac{1}{\mu L''(p)}$$

$F^{-1}(p)$ denotes the inverse function of $F(x)$, thus, Lorenz curve can be written as:

$$L(p) = \frac{1}{\mu}\int_0^p F^{-1}(q)\mathrm{d}q$$

We call $L(p)$ with $p \in [0,1]$ a Lorenz curve if it satisfies[3]:

$$L(0) = 0, L(1) = 1, L^{'}(p) \geq 0, L^{''}(p) \geq 0$$

(1)

As is known, Lorenz curve can better reflect the inequality of the income distribution, and there has been a lot of studies on it, some Lorenz curve models have been applied to produce approximate Lorenz curve fitting. According to relative literature, 13 Lorenz curve models are summed up in table 1[3-12].

Table 1. 13 Lorenz curve models

| Lorenz curves | parameters |
|---|---|
| $L(P) = P^{\alpha}$ | $\alpha > 1$ |
| $L(P) = Pa^{p-1}$ (1) | $\alpha > 0$ |
| $L_3(p) = \frac{(1-\beta)p}{1-\beta p}$ | $0 \leq \beta \leq 1$ |
| $L_4(P) = p(k+(1-k)p)$ | $0 \leq k < 1$ |
| $L_5(p) = pe^{-\delta(1-p)}$ | $\delta > 0$ |
| $L_6(p) = 1-(1-p)^{\delta}$ | $0 < \delta < 1$ |
| $L_7(p) = 1-(1-p^k)^{1/k}$ | $k > 1$ |
| $L_8(p) = \frac{e^{kp-1}}{e^k - 1}$ (3) | $k > 0$ |
| $L_9(p) = \frac{p(1-2\theta)(1+\theta p)}{(1+\theta)(1-2\theta p)}$ | $0 \leq \theta \leq \frac{1}{2}$ |
| $L_{10}(p) = p^{\alpha}\left[1-(1-p)^{\beta}\right]$ (2) | $\alpha \geq 0, 0 < \beta \leq 1$ |
| $L_{11}(p) = p^{\alpha}e^{-\eta(1-p)}$ (4) | $1 < \alpha < 2, \eta > 0$ |
| $L_{12}(p) = \left[1-(1-p)^{\alpha}\right]^{1/\beta}$ | $0 < \alpha \leq 1, 0 < \beta \leq 1$ |
| $L_{13}(p) = \frac{(\beta-1)p^{\alpha+1}}{\beta - P}$ | $\alpha > 1, 0 < \beta < 1$ |

Based on the nature of the Lorenz curve and the existing literature, a new Lorenz curve model is built as follows:

$$L_{14}(p) = 1-(1-P^{\alpha})^{\beta} \tag{2}$$

Where α≥1, 0≤β≤1.

It can be proved that the new model satisfies the definition of Lorenz curve.

**Proof:**

When p∈[0,1], L(p)∈[0,1],

$$L_{14}(0) = 0\,, L_{14}(1) = 1$$

$$\begin{aligned} L_{14}^{'}(p) &= -\beta(1-p^{\alpha})^{\beta-1} \\ &= \alpha\beta(1-p^{\alpha})^{\beta-1} p^{\alpha-1} \end{aligned}$$

When α>0, β>0 and p∈ [0,1], $L_{14}^{`}$(p) ≥0;

$$\begin{aligned} L_{14}^{''}(p) &= \alpha\beta[(\beta-1)(1-p^{\alpha})^{\beta-2}(-\alpha p^{\alpha-1})p^{\alpha-1} \\ &+(1-p^{\alpha})^{\beta-1}(\alpha-1)p^{\alpha-2}] \\ &= \alpha^{2}\beta(1-\beta)(1-p^{\alpha})^{\beta-2}p^{2\alpha-2}+\alpha(\alpha-1)\beta(1-p^{\alpha})^{\beta-1}p^{\alpha-2} \end{aligned}$$

When α≥1, 0≤β≤1, and p∈[0,1], $L_{14}^{``}$(p) ≥0.

Thus, the new model satisfies the definition and nature of Lorenz curve. Then we compare Model 14 and model 1-13.

The kernel estimation can be used to estimate the income-distribution based on the income and consumption data at the household level. The complete form of this kind of data is:

$$(p_i, x_i/\mu), \quad i = 1,2,\cdots,n \tag{3}$$

$$(p_i, L_i), \quad i = 1,2,\cdots,n \tag{4}$$

$x_i$ is the interval point, which satisfies $0 \le x_1 < x_2 < \ldots < x_n < x_{n+1}$, then the proportion of population in *[xi,xi+1)* is $p_i$-$p_{i-1}$. If *l(p)* is the real Lorenz curve of income distribution and *l'(p)* exists, then (3) represents a coordinate point on *l'(p)*, namely, $l'(p_i)=x_i/\mu$; (4) represents a coordinate point on *l(p)*, namely, $l(p_i)=L_i$. We use the Lorenz curve model $L(p, \tau)$ in economics to fit the data in (4), among which $\tau$ is a set of parameters.

Use the nonlinear least square method to solve:

$$\min \sum_{i=1}^{n} \left( L(p_i, \tau) - L_i \right)^2$$

(5)

And we can obtain the estimated value $\hat{\tau}$ of the parameter vector $\tau$, then, $L(p,\hat{\tau}) = \hat{L}(p)$ can be used as the approximate Lorenz curve to analyze the income distribution.

To compare the fitting precision, the following three criteria are used:

MSE, mean squared error:

$$\frac{1}{n}\sum_{i=1}^{n}\left[L\left(p_i, \hat{\tau}\right) - L_i\right]^2$$

MAE, mean absolute error:

$$\frac{1}{n}\sum_{i=1}^{n}\left|L\left(p_i, \hat{\tau}\right) - L_i\right|$$

MAS, maximum absolute error:

$$\max_{1 \le i \le n}\left|L\left(p_i, \hat{\tau}\right) - L_i\right|$$

The above 13 models and the model 14 are applied to the datasets and the estimated errors are displayed in Table 2.

Table 2. Estimation errors of 14 Lorenz curve models

| Models | Parameters | MSE | MAE | MAS |
|---|---|---|---|---|
| $L_1(P) = P^{\alpha}$ | $\alpha$=2.390413 | 0.0006870311 | 0.02134044 | 0.05634371 |
| $L_2(P) = Pa^{p-1}$ | $\alpha$=6.012497 | 0.0002660086 | 0.01309733 | 0.03951285 |
| $L_3(p) = \dfrac{(1-\beta)p}{1-\beta p}$ | $k$=0.7123623 | 0.0005000651 | 0.01999309 | 0.03286169 |
| $L_4(P) = p(k+(1-k)p)$ | $k$=0.1829778 | 0.001191501 | 0.02881912 | 0.06901038 |
| $L_5(p) = pe^{-\delta(1-p)}$ | $\delta$=1.79384 | 0.0002660086 | 0.01309733 | 0.03951285 |
| $L_6(p) = 1-(1-p)^{\delta}$ | $\delta$=0.4641201 | 0.003051689 | 0.04960815 | 0.07511602 |
| $L_7(p) = 1-(1-p^k)^{1/k}$ | $k$=1.563489 | 0.0002645976 | 0.01459461 | 0.02255173 |
| $L_8(p) = \dfrac{e^{kp-1}}{e^k-1}$ | $k$=2.689169 | 0.0002738314 | 0.01408636 | 0.03588281 |
| $L_9(p) = \dfrac{p(1-2\theta)(1+\theta p)}{(1+\theta)(1-2\theta p)}$ | $\theta$=0.3344691 | 0.0003557203 | 0.01701333 | 0.02627239 |
| $L_{10}(p) = p^{\alpha}\left[1-(1-p)^{\beta}\right]$ | $\alpha$=0.8031218<br>$\beta$=0.6794427 | 9.383024e-06 | 0.002741164 | 0.005523377 |
| $L_{11}(p) = p^{\alpha}e^{-\eta(1-p)}$ (4) | $\alpha$=1.105312<br>$\eta$=1.655577 | 0.0002617151 | 0.01230699 | 0.04088574 |

| | | | | |
|---|---|---|---|---|
| $L_{12}(p)=\left[1-(1-p)^{\alpha}\right]^{1/\beta}$ (5) | $\alpha$=0.7819596 $\beta$=0.5672266 | 1.82089e-05 | 0.003770052 | 0.007969309 |
| $L_{13}(p)=\dfrac{(\beta-1)p^{\alpha+1}}{\beta-P}$ | $\beta$=0.4276119 $\alpha$=1.6421735 | 0.0001801164 | 0.01051966 | 0.03306539 |
| $L_{14}(p)=1-(1-P^{\alpha})^{\beta}$ | $k$=1.8525579 $\alpha$=0.7489707 | 9.026087e-06 | 0.002655113 | 0.005569565 |

Through the comparison table of fitting precision, it is obvious that the former 13 models is inferior to model 14. According to the standard of MSE, Model 14 is the best. According to the standard of MAE, Model 14 is also the best. According to the standard of MAS, Model 14 is only inferior to Model 10. Overall, the fitting precision of Model 14 is better than most of the compared Lorenz curve models, and the goodness of fit is almost 1. Thus, the new proposed Model 14 is an attractive Lorenz curve model.

## 3. The improved income space method

Middle-income population is an important index and measure of income distribution pattern. Under the ideal featured pattern of income distribution, the income gap is not big, the social consumption is vigorous, people’s living level is high, and thus the society is stable. Income space method is at present a common method used by the economic theoretical circle to define the middle-income population. As is shown in figure 3, taking an interval ($x_l$, $x_h$) within a range of median income $m$ as middle-income population, then the middle income ratio equals to F($x_h$)-F($x_l$).

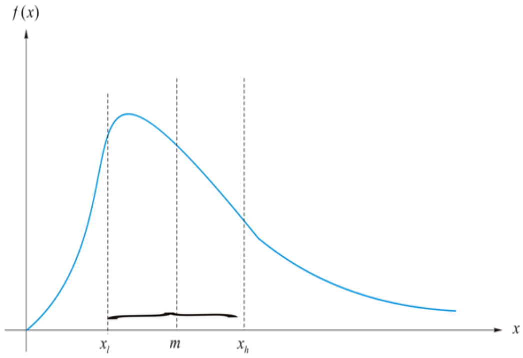


Fig.3. Income distribution density function

Obviously, in this method, $x_l$ and $x_h$ can be chosen arbitrarily. But the interval of income varies due to the improvement of economy or inflation, a more common situation may be that all people’s income have been improved, that is to say, the income range of the whole society moves to the right. Thus, vertically, it will be difficult to compare the middle-income population of different years.

The defects of this method are discussed in [13-16], which points out the arbitrariness of this definition, for example, why choose interval of 75%-125% of the median income instead of 60%-225%. Thus, it is of great practical significance to build a reasonable dynamic interval to describe middle income population.

To work out the arbitrariness of choosing $x_l$, $x_h$, a specific improved method according to the entire fairness and local fairness of the whole society is established. Thus, the middle-income population can be defined and compared.

First, define local fair index $D = \frac{L'(p)}{L''(p)}$, then explain why it is called that. In *[p, p+Δp]*, the sharper *L'(p)* changes, the greater the difference of contribution of different quantile's people is, and that is to say, the people's income in this interval is unfair. Thus, $\frac{L'(p+\Delta p)}{L'(p)}$ can be used to measure the fairness degree in [$p$, $p+\Delta p$], the greater it is, the more unfair the society will be. (it can also be got from the figure, the initial derivative values are the same, but the derivative value of the solid curve is bigger than that of the virtual curve, thus, the virtual curve is more protruding than the solid curve and $\frac{L'(p+\Delta p)}{L'(p)}$ can be used to measure the degree of the lower convex of the curve.).

Hence, the bigger $\frac{L'(p+\Delta p)}{L'(p)}$ is, the more convex the *Lorentz* curve will be, thereby, the more unfair the social distribution will be. Allow for $\ln(\frac{L'(p+\Delta p)}{L'(p)}) = \ln L'(p+\Delta p) - \ln L'(p) = \int_p^{p+\Delta p} \frac{L''(t)}{L'(t)} dt$, $\frac{L''(p)}{L'(p)}$ can be taken as the unfair rate of a society, then, its reciprocal $D(p) = \frac{L'(p)}{L''(p)}$ is the fair rate of a society, or is called local fair index. The bigger the value is, the fairer the society will be. Meanwhile, a closely related *Gini* coefficient in economics is used to measure the polarization, and its definition is two times the area between *Lorentz* curve and 45° line.

*Gini* coefficient: $G = 1 - 2\int_0^1 L(p)dp$.

When the society attaches entire fairness, the *Lorentz* curve is $L(p)=p$, $L''(p)=0$, $\frac{1}{L''(p)} = \infty$.

$$G = 2\int_0^1 (p - L(p))dp = \frac{\int_0^1 (p - L(p))dp}{\int_0^1 p dp} \tag{6}$$

It actually is the ratio of the area between *Lorentz* curve and 45° line and the area of the whole triangle under the diagonal, it measures the fairness degree of the whole society, the greater the *Gini* index is, the more unfair the society will be. In order to further study the problem, it is necessary to study the fairness degree of the less than middle-income population and more than middle income population. Similarly, so an augmented *Gini* index is defined as follows:

$$G_1 = \frac{\int_0^{\frac{1}{2}} (p - L(p))dp}{\int_0^{\frac{1}{2}} pdp} = 8\int_0^{\frac{1}{2}} (p - L(p))dp \tag{7}$$

$$G_2 = \frac{\int_{\frac{1}{2}}^{1} (p - L(p))dp}{\int_{\frac{1}{2}}^{1} pdp} = \frac{8}{3}\int_{\frac{1}{2}}^{1} (p - L(p))dp$$

(8)

The relationship between them is: $G = \frac{1}{4}G_1 + \frac{3}{4}G_2$.

Therefore, the income space method is improved and the income range of middle income population is got:

$$[(1-\frac{1}{2}D(1-G_1))m, (1+\frac{1}{2}D(1-G_2))m]$$

Since $f(x) = \frac{1}{uL''(p)}$, so when $p = \frac{1}{2}$, the corresponding *f(x)* is the population density near the median m. since $L'(\frac{1}{2}) = \frac{m}{u}$, so $D = \frac{L'(\frac{1}{2})}{L''(\frac{1}{2})} = \frac{m}{u}\frac{1}{L''(\frac{1}{2})} = mf(m)$.

Actually, it is the income intensity of the population near the median *m*. The greater the intensity is, the higher the upheaval in the middle of the figure will be. That is to say, *D* will become bigger, and the income range of the middle income population will be larger, therefore, the proportion of middle income population will be larger. This is consistent with the words "if the middle section uplift even higher than the previous year, then the middle-income population expanded"; meanwhile, when the population on both sides expands, the polarization will be severe, then $G, G_1, G_2$ becomes larger, the income range of middle income population will be smaller. This is also consistent with the principle "if the population of both sides expands, then the middle-income population decreases". With the development of economy, the defined range of middle-income population varies according to the improvement of middle income *m*.

The middle income population is always selected proportionally in economics, but how much percentage should be chosen is unwarranted according to the method. In this paper, we build a new model based on the social fairness, thus, not only can we determine the proportion, but also, we can well explain the principle of economics.

## 4. Results and discussion

The method proposed in this paper is fitted to a dada offered by the Problem E in the National Graduate Mathematical Contest in Modeling in 2013. The data comprises of the grouped data of the income distribution of 2 different years in 2 different regions. Based on the improved income space method, we use Model 14: $L(p)=1-(1-p^{\alpha})^{\beta}$ to make quantitative analysis of different years of region A, B.

### 4.1 Vertical comparison

Seen from table 3, the average income and median income of year 2 are improved compared with year 1 in region *A*, but the Gini coefficient increases, thus, the polarization is severe, the range of middle-income quantile shrinks, and the proportion of middle-income population decreases.

This can also be seen from figure 4, compared with year 1, the tail of the density function of year 2 becomes longer and thicker. Thus, the polarization becomes severe, and this is consistent with the increase of *Gini* coefficient. At the same time, the height of middle-income changes from upheaval to flat, which denotes the decrease of the proportion of middle-income population and the right shift of the income range of middle-income population. This is identical to the result of the model.

So, our model is well consistent with the actual situation.

Table 3. 2 years of region *A*

| Different times of region *A* | Year 1 | Year 2 |
|---|---|---|
| Average income | 6281.34 | 8890.21 |
| Median income | 5254.75 | 7461.83 |
| *Gini* coefficient | 0.3423 | 0.3476 |
| Income range of middle-income population | [4516.30, 6565.80] | [6465.53, 9264.96] |
| Quantile range of middle-income population | [0.3856, 0.6343] | [0.3963, 0.6301] |
| Proportion of middle-income population | 24.87% | 23.28% |

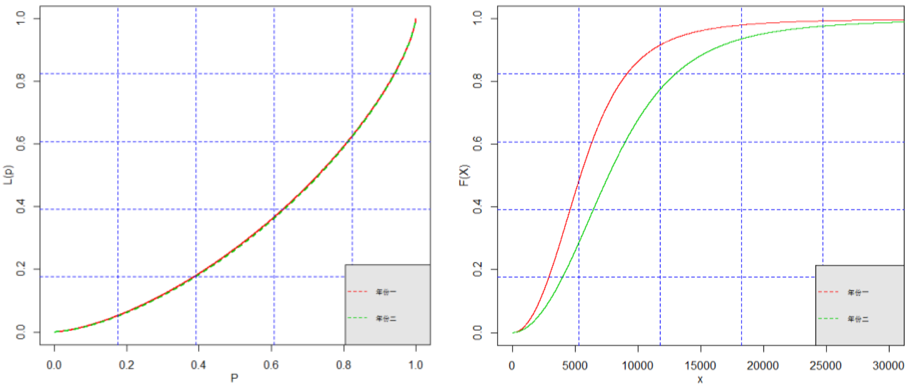

Fig.4. The comparison of different years in region *A*

Seen from table 4, the average income and median income of year 2 is improved compared with year 1 in region *B*, but the *Gini* coefficient decreases, the polarization is slightly lightened, the range of middle income quantile shrinks and move to the right, and the proportion of middle income population decreases. This is slightly different from region *A*, although the polarization of year 2 decreases, the crowd density of middle income is sparse; while the polarization of year 1 is slightly bigger than year 2, but the crowd density of middle income is strong enough to offset the increased polarization, so year 1 has a higher proportion of middle income population.

This can also be seen from figure 5, compared with year 1, the height of middle income of year 2 changes from upheaval to flat. This denotes the decrease of the proportion of middle income population and the right shift of the income range of middle income population. This is identical to the result of the model.

Table 4. 2 years of region *B*

| Different times of region *B* | Year 1 | Year 2 |
|---|---|---|
| Average income | 16938.46 | 22228.53 |
| Median income | 14422.27 | 19814.62 |
| *Gini* coefficient | 0.2713 | 0.2575 |
| Income range of middle-income population | [10580.31, 19912.01] | [14719.47, 27373.56] |
| Quantile range of middle-income population | [0.2117, 0.7562] | [0.2374, 0.7625] |
| Proportion of middle-income population | 54.45% | 52.51% |

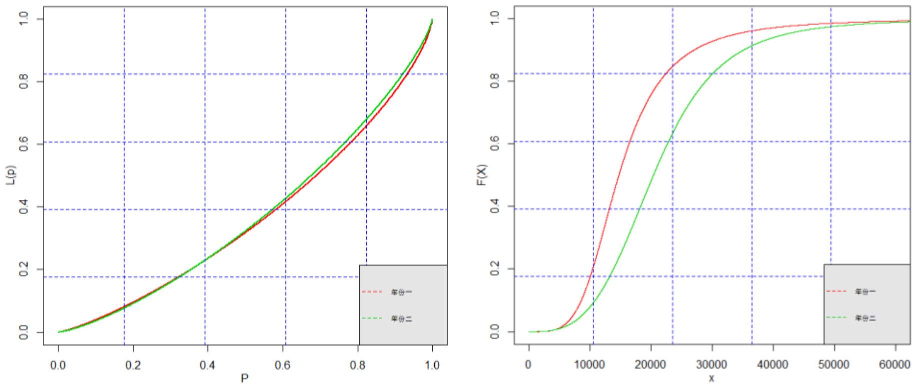


Fig.5. The comparison of different years in region B

Considering the changes of the data, the increasing rate of average income in region $A$: (8890.21-6281.34)/6281.34=41.53%, the increasing rate of average income in region *B*: (22228.53-16938.46)/16938.46=31.23%, the decrement rate of the proportion of middle-income population in region *A*: (23.28%-24.87%)/24.87%=5.57%,The decrement rate of the proportion of middle-income population in region *B*: (52.51%-54.45%)/54.45%=4.4%.

The *Gini* coefficient of region *A* increases, while the *Gini* coefficient in region *B* decreases. Thus, we can see that: the increasing rate of the average income in region *A* is higher than that in region *B*, but it is at the expense of the more decrease of middle-income population. Therefore, the *Gini* coefficient increases, and the polarization becomes severe.

## 4.2 Horizontal comparison

Table 5. 2 different regions of Year 1

| Different regions of year 1 | *A* | *B* |
|---|---|---|
| Average income | 6281.34 | 16938.46 |
| Median income | 5254.75 | 14422.27 |
| *Gini* coefficient | 0.3423 | 0.2713 |
| Income range of middle income population | [4516.30, 6565.80] | [10580.31, 19912.01] |
| Quantile range of middle income population | [0.3856, 0.6343] | [0.2117, 0.7562] |
| Proportion of middle income population | 24.87% | 54.45% |

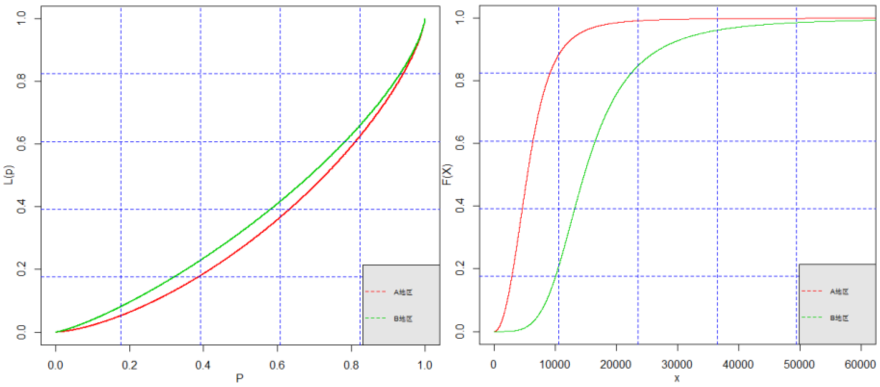

Fig.6. The comparison of different regions of year 1

Seen from table 5, no matter what index, *A* is behind *B*: the average income, middle income, income range of middle income population, range of quantile and proportion of middle income population of *A* is far below *B*; but the *Gini* coefficient of *A* is bigger than *B*, meanwhile, the polarization of *A* is more severe than *B*.

This can also be seen from figure 6, the middle income range of region *A* is almost all on the left of *B*, that is to say, the middle-income population of region *B* can be seen as high-income in region *A*. All in all, the economy in region *A* is far behind region *B*.

Table 6. 2 different regions of Year 2

| Different regions of year 2 | *A* | *B* |
|---|---|---|
| Average income | 8890.21 | 22228.53 |
| Median income | 7461.83 | 19814.62 |
| *Gini* coefficient | 0.3476 | 0.2575 |
| Income range of middle income population | [6465.53, 9264.96] | [14719.47, 27373.56] |
| Quantile range of middle income population | [0.3963, 0.6301] | [0.2374, 0.7625] |
| Proportion of middle income population | 23.28% | 52.51% |

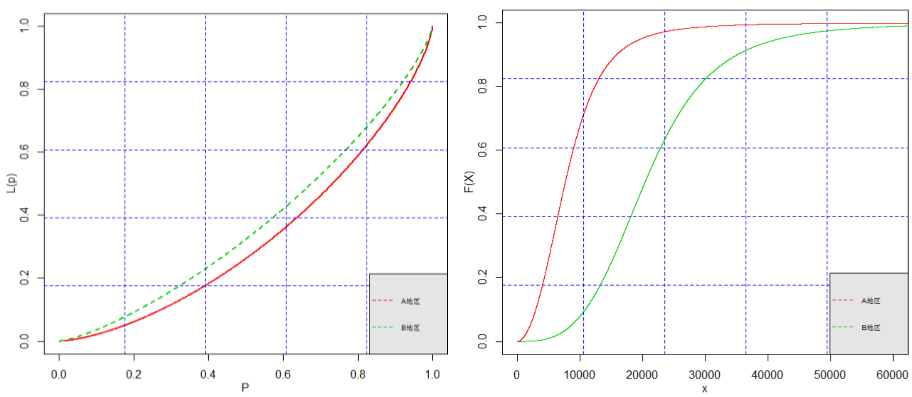

Fig.7. The comparison of different regions of year 2

Similarly, no matter what index, *A* is still behind *B*, that is to say, the economy in region *A* is behind region *B*.

All in all, horizontally, the economy of region *B* is obviously better than region *A*. Vertically, the improvement of economy in region *A* becomes fast, but it can not be regarded as a good form. Actually, this development pattern is unsustainable.

## 5. Conclusion

A new Lorenz curve model is proposed and fitted with the given data, the results of the analysis show that the accuracy of the new model is superior to most of the existing models, which proves that the new model is an attractive one. Then, based on the curvatures of the curve, an improved income space method is put forward. Finally, the new Lorenz model and the improved income method, which denote the fairness of income distribution, are applied to analyze the data from Problem E in the National Graduate Mathematical Contest in Modeling in 2013, the results of this test suggest that the proposed Lorenz model performs very well for the datasets.

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