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A New Similarity Measure of Intuitionistic Fuzzy Set and Application in MADM Problem

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Abstract

Similarity measure is an important tool to measure the degree of resemblance between two intuitionistic fuzzy sets. In this paper, in order to overcome the counter-intuitive in some cases, a new similarity measure of intuitionistic fuzzy sets is constructed and successively applied in pattern recognition and medical diagnosis. Based on the proposed similarity measure, a new decision making method is put forward for the multi-attribute decision making (MADM) problem with attribute values expressed by intuitionistic fuzzy set. When the attribute weights information is completely unknown, maximizing deviation method is developed and used to determine the weights. When the attribute weights information is partly known, an optimization model is established for solving the attribute weights. Two MADM examples are given to illustrate the feasibility and practicability of the proposed decision making method.

Key words Similarity measure, intuitionistic fuzzy set, multi-attribute decision making method, TOPSIS

1. Introduction

Fuzzy set was firstly proposed by Zadeh [1], and has been extensively studied and applied to many fields, such as decision making, medical diagnosis. Though fuzzy sets have gained great success, but in some cases, especially when the decision maker wants to express his/her hesitation about some evaluation attribute values, fuzzy sets may not be a suitable tool. Intuitionistic fuzzy set (IFS), proposed by Atanassov [2], can well model this situation by adding a non-membership degree. The applications of IFS are permeated into many fields, including decision making problems [3-10], medical diagnosis [11, 12], pattern recognitions [13-15] and image processings [16, 17]. Similarity measure is an important tool for measuring the degree of similarity between two IFSs. Many similarity measures of IFSs are investigated in the literature [18, 19]. Li and Chen [20] firstly gave the definition of the similarity measure between IFSs, and they also proposed some similarity measures simultaneously applying them in pattern recognition. Li and Chen's similarity measure takes into the medians of two intervals only, and thus it can easily be pointed out the counter-intuitive examples, then Liang and Shi [21] put forward some more reasonable similarity measures through numerical comparisons with Li and Chen's similarity measures. Mitchell [22] proposed an improved similarity from a statistical viewpoint on the basis of Li and Chen's similarity measure. Some similarity measures have been constructed based on distance measures, such as Szmidt and Kacprzyk [23] constructed similarity measures using the Hamming distance measure and put them into the multi-attribute group decision making problem. Hung and Yang [24] constructed similarity measure using Hausdorff distance. Hung and Yang [25] induced similarity measures using L_p measure. Xu and Chen [26] gave comprehensive overview and comparison of distance and similarity measures between IFSs. Ye [27] proposed a cosine measure for IFSs using cosine function. To overview the prior-proposed similarity, Baccour et al. [19] summarized the existed similarity measures and pointed out that each above similarity measure has drawbacks. More recently, Hwang and Yang [28] given a new construction for similarity measures for IFSs by defined lower, upper and middle fuzzy sets, and they found that the new constructed similarity measures can improve most existing similarity measures.

The main aim of this paper is to put forward a similarity measure for IFSs, which can better measure the degree of similarity measure between IFSs. Because there are still drawbacks on the ranking function and operation rules about IFSs [29], the similarity measure can be used to overcome these shortcomings. Thus this paper will develop a new decision making method based on the proposed similarity measure for the MADM under intuitionistic environment with attribute weights information is partly known.

The rest of this paper is organized as follows: In Section 2, we first briefly recall the definition and similarity measure of IFSs, and then we review the several existing similarity measures. In Section 3, we construct a new similarity measure of IFSs, and compare it with other similarity measures through some examples. In Section 4, based on the new similarity measure and TOPSIS method, we put forward a new decision making method for the MADM problem in which attribute values expressed by intuitionistic fuzzy numbers and attribute weights are partly known. Section 5 gives the discussion of the proposed method through two examples. Finally, the conclusions are given in Section 6.

2. Intuitionistic Fuzzy Set and Similarity Measures

In this section we firstly review the IFS and similarity measures given by other authors.

Definition 1 [2]. Let $X=\{x_1,x_2,...,x_n\}$ be the universe of discourse, $A=\{(x,\mu_A(x),\upsilon_A(x))|x\in X\}$ is called an IFS in X, where $\mu_A(x):X\rightarrow[0,1]$ and $\upsilon_A(x):X\rightarrow[0,1]$ are the membership degree and non-membership degree functions of x belonging to A, respectively, and they satisfy $0\le\mu_A(x)+\upsilon_A(x)\le 1$ for $\forall x\in X$. Each element $(x,\mu_A(x),\upsilon_A(x))$ of A is called intuitionistic fuzzy number (IFN), and often briefly noted by $(\mu_A(x),\upsilon_A(x))$.

Definition 2 [2]. For each IFN $(x,\mu_A(x),\upsilon_A(x))$, $\pi_A(x)=1-\mu_A(x)-\upsilon_A(x)$ is called the hesitancy degree, which can express the hesitancy degree of decision maker. Obviously, $0\le\pi_A(x)\le 1$.

Definition 3[2,30]. Let $A=\{(x_i,\mu_A(x_i),\upsilon_A(x_i))|x_i\in X\}$ and $B=\{(x_i,\mu_B(x_i),\upsilon_B(x_i))|x_i\in X\}$ be two IFSs, then

(1) $A\subseteq B$ if and only if $\mu_A(x_i)\le\mu_B(x_i),\upsilon_A(x_i)\ge\upsilon_B(x_i),\forall x_i\in X$;

(2) $A=B$,if and only if $A\subseteq B$ and $B\subseteq A$, i.e. $\mu_A(x_i)=\mu_B(x_i),\upsilon_A(x_i)=\upsilon_B(x_i),\forall x_i\in X$;

(3)The complementary set of A noted by A^C is $A^C=\{<x_i,\upsilon_A(x_i),\mu_A(x_i)>|x_i\in X\}$

In the following discussion, we always use $IFSs(X)$ denote the set of all IFSs in X. The Definition 4 will introduce the definition of similarity measure between two IFSs A and B.

Definition 4 [18]. Let A and B be two IFSs, and S is a mapping $S: IFSs(X)\times IFSs(X)\rightarrow[0,1]$. Then $S(A,B)$ is called the similarity measure between A and B if it satisfies the following conditions:

(i) $0\le S(A,B)\le 1$

(ii) $S(A,B)=1$ if $A=B$;

(iii) $S(A,B)=S(B,A)$

(iv) If $A\subseteq B\subseteq C$, then $S(A,C)\le\min\{S(A,B),S(B,C)\}$.

(v) $S(A,A^C)=0$ if and only if A is a crisp set.

In the following, we will give a brief review of existing similarity measures of IFSs. Considering two IFSs $A=\{(x_i,\mu_A(x_i),\upsilon_A(x_i))\,|\,x_i\in X\}$ and $B=\{(x_i,\mu_B(x_i),\upsilon_B(x_i))\,|\,x_i\in X\}$, the existing similarity measures are reviewed as follows:

(1) Chen's similarity measure is defined as [31]

$$S_C(A,B)=1-\frac{1}{2n}\sum_{i=1}^{n}|S_A(x_i)-S_B(x_i)|$$

where $S_A(x_i)=\mu_A(x_i)-\upsilon_A(x_i)$ and $S_B(x_i)=\mu_B(x_i)-\upsilon_B(x_i)$.

But, as Hong and Kim [31] noticed that when $S_A(x_i)=S_B(x_i)$, $S_C(A,B)=1$,which is counter-intuitive, e.g., for $A_0=(x,0,0)$ and $B_0=(x,0.5,0.5)$, we have $S_C(A_0,B_0)=1$.

(2) Hong and Kim's similarity measures are defined as [32]

$$S_H(A,B)=1-\frac{1}{2n}\sum_{i=1}^{n}(|\mu_A(x_i)-\mu_B(x_i)|+|\upsilon_A(x_i)-\upsilon_B(x_i)|)$$

$$S_L(A,B)=1-\frac{1}{4n}\sum_{i=1}^{n}(|S_A(x_i)-S_B(x_i)|+|\mu_A(x_i)-\mu_B(x_i)|+|\upsilon_A(x_i)-\upsilon_B(x_i)|)$$

Since $S_H(A,B)$ and $S_L(A,B)$ only take into account the absolute values, it does not distinguish the positive from negative differences, e.g., for $A_1=(x,0.3,0.3)$, $B_1=(x,0.4,0.4)$, $C_1=(x,0.3,0.4)$ and $D_1=(x,0.4,0.3)$, we have $S_H(A_1,B_1)=S_H(C_1,D_1)=0.9$, and for $A_2=(x,0.4,0.2)$, $B_2=(x,0.5,0.3)$, $C_2=(x,0.5,0.2)$, we have $S_L(A_2,B_2)=S_L(A_2,C_2)=0.95$ which are also counter-intuitive.

(3) Li et al.'s similarity measure is defined as [33]

$$S_O(A,B)=1-\left(\frac{1}{2n}\sum_{i=1}^{n}((\mu_A(x_i)-\mu_B(x_i))^2+(\upsilon_A(x_i)-\upsilon_B(x_i)^2)\right)^{1/2}$$

The same problem like with $S_H(A,B)$ occurs with the similarity measure $S_O(A,B)$ due to the result of $S_O(A_1,B_1)=S_O(C_1,D_1)=0.9$.

(4) Li and Chen's similarity measure is defined as [20]

$$S_d^p(A,B)=1-\frac{1}{\sqrt[n]{p}}\sqrt[p]{\sum_{i=1}^{n}|m_A(x_i)-m_B(x_i)|^p}$$

Where $m_A(x_i)=\dfrac{\mu_A(x_i)+1-\upsilon_A(x_i)}{2}$, $m_B(x_i)=\dfrac{\mu_B(x_i)+1-\upsilon_B(x_i)}{2}$ and $1\le p<+\infty$.

The similarity measure takes into the medians of two intervals only, and thus we can easily point out the counter-intuitive examples, e.g., $A_2=(x,0.4,0.2)$, $B_2=(x,0.5,0.3)$, we have $S_d^p(A_2,B_2)=1$ for each p.

(5) Mitchell's similarity measure is defined as [22]

$$S_{HB}(A,B)=\frac{1}{2}(\rho_\mu(A,B)+\rho_\upsilon(A,B))$$

Where $\rho_\mu(A,B)=1-\frac{1}{\sqrt[n]{p}}\sqrt[p]{\sum_{i=1}^{n}|\mu_A(x_i)-\mu_B(x_i)|^p}$ and $\rho_\upsilon(A,B)=1-\frac{1}{\sqrt[n]{p}}\sqrt[p]{\sum_{i=1}^{n}|\upsilon_A(x_i)-\upsilon_B(x_i)|^p}$.

Mitchell [22] modified Li and Chen's similarity measure using a statistical approach by interpreting IFSs as families of ordered fuzzy sets. Unfortunately, when $p=1$, $S_{HB}(A,B)$ is equal to $S_H(A,B)$, and thus has the same counter-intuitive results as $S_H(A,B)$.

(6) To overcome the drawbacks of Li and Chen's similarity measure, Liang and Shi [21] also proposed three similarity measures as follows:

(i) $$S_e^p(A,B)=1-\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}(\varphi_{\mu AB}(x_i)+\varphi_{\upsilon AB}(x_i))^p}$$

where $\varphi_{\mu AB}(x_i)=\frac{|\mu_A(x_i)-\mu_B(x_i)|}{2}$, $\varphi_{\upsilon AB}(x_i)=\frac{|(1-\upsilon_A(x_i))-(1-\upsilon_B(x_i))|}{2}$ and $1\le p<+\infty$.

But, for $S_e^p(A,B)$,when $p=1$, $S_e^p(A,B)=S_{HB}(A,B)=S_H(A,B)$, and thus has the same counter-intuitive results as $S_H(A,B)$.

(ii) $$S_s^p(A,B)=1-\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}(\varphi_{s1}(x_i)+\varphi_{s2}(x_i))^p}$$

where $\varphi_{s1}(x_i)=\frac{|m_{A1}(x_i)-m_{B1}(x_i)|}{2}, \varphi_{s2}(x_i)=\frac{|m_{A2}(x_i)-m_{B2}(x_i)|}{2}$

$m_{A1}(x_i)=\frac{|\mu_A(x_i)-m_A(x_i)|}{2}, m_{A2}(x_i)=\frac{|m_A(x_i)+1-\upsilon_A(x_i)|}{2}$,

$m_A(x_i)=\frac{\mu_A(x_i)+1-\upsilon_A(x_i)}{2}$, and $m_B(x_i)=\frac{\mu_B(x_i)+1-\upsilon_B(x_i)}{2}$.

Though $S_s^p(A,B)$ avoids the problematic results obtained from $S_d^p(A,B)$ when the IFSs have equal medians, it still has counter-intuitive results in some cases. For example, for $A_2=(x,0.4,0.2)$, $B_2=(x,0.5,0.3)$, $C_2=(x,0.5,0.2)$, we have $S_s^p(A_2,B_2)=S_s^p(A_2,C_2)=0.95$, which also seems to be not accordance with the fact.

(iii) $$S_h^p(A,B)=1-\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}(\sum_{m=1}^{3}w_m\eta_m(x_i))^p}$$

where $\eta_1(x_i)=\frac{\varphi_{s1}(x_i)+\varphi_{s2}(x_i)}{2}, \eta_2(x_i)=|m_A(x_i)-m_B(x_i)|, \eta_3(x_i)=\max\{l_A(x_i),l_B(x_i)\}-\min\{l_A(x_i),l_B(x_i)\}$,

$l_A(x_i)=\frac{1-\upsilon_A(x_i)-\mu_A(x_i)}{2}$, and $l_B(x_i)=\frac{1-\upsilon_B(x_i)-\mu_B(x_i)}{2}$.

The counter-intuitive occurs under the case of $S_h^p(A_1,B_1)=S_h^p(C_1,D_1)=0.933$, which has the same drawback with $S_H(A,B)$ and $S_O(A,B)$.

(7) Hung and Yang's similarity measures $S_{HY}^k(A,B)$ $(k=1,2,3)$ are defined as [24]

$S_{HY}^1(A,B)=1-d_H(A,B)$, $S_{HY}^2(A,B)=\dfrac{e^{-d_H(A,B)}-e^{-1}}{1-e^{-1}}$ and $S_{HY}^3(A,B)=\dfrac{1-d_H(A,B)}{1+d_H(A,B)}$,where $d_H(A,B)$ is the Hausdorff distance measure, i.e.

$$d_H(A,B)=\sum_{i=1}^{n}\max\{|\mu_A(x_i)-\mu_B(x_i)|,|\upsilon_A(x_i)-\upsilon_B(x_i)|\}.$$

Unfortunately, we have the results $S_{HY}^1(A_1,B_1)=S_{HY}^1(C_1,D_1)=0.9$, $S_{HY}^2(A_1,B_1)=S_{HY}^2(C_1,D_1)=0.85$ and $S_{HY}^3(A_1,B_1)=S_{HY}^3(C_1,D_1)=0.82$ which imply that Hung and Yang's similarity measures also have counter-intuitive result.

(8) Hwang and Yang's similarity measures [28] are defined based on the lower, upper and middle fuzzy sets which is a new construction method. The new constructed similarity measures improve the original similarity measures, but there still exists counter-intuitive result. For example, the following similarity

$$S(A,B)=1-\frac{1}{3n}\sum_{i=1}^{n}(\mu_A(x_i)-\mu_B(x_i)|+|\upsilon_A(x_i)-\upsilon_B(x_i)|+\frac{1}{2}|\mu_A(x_i)-\mu_B(x_i)\text{-}\upsilon_A(x_i)+\upsilon_B(x_i)|)$$

is a similarity measure satisfying the constructing rule of Hwang and Yang's similarity measures [28]. Considering with the IFSs $A_3=(x,0.2,0.6)$, $B_3=(x,0.3,0.5)$ and $C_3=(x,0.1,0.7)$. From the perspective of closer to against the evidence, similarity measure $S(A_3,C_3)$ should bigger than $S(A_3,B_3)$. But the Hwang and Yang's similarity measures all get the result $S(A_3,C_3)=S(A_3,B_3)$.

(9) Ye's cosine similarity measure defined as [27]

$$C_{IFS}(A,B)=\frac{1}{n}\sum_{i=1}^{n}\frac{\mu_A(x_i)\mu_B(x_i)+\upsilon_A(x_i)\upsilon_A(x_i)}{\sqrt{\mu_A^2(x_i)+\mu_B^2(x_i)}\sqrt{\upsilon_A^2(x_i)+\upsilon_B^2(x_i)}}$$

The similarity measure still has counter-intuitive example, such as $A_4=(x,0.1,0.1), B_4=(x,0.4,0.4)$, we have $C_{IFS}(A,B)=1$.

Remark 1. From above analysis, we see that the existing similarity measures have counter-intuitive cases, and thus in the next section, we will put forward a better similarity measure which can overcome the counter-intuitive cases.

3. A New Effective Similarity Measure Proposed

To avoid counter-intuitive phenomena, a new effective similarity measure of IFSs is induced in the follows.

Definition 5. Assume that $A=\{(x_i,\mu_A(x_i),\upsilon_A(x_i))\,|\,x_i\in X\}$ and $B=\{(x_i,\mu_B(x_i),\upsilon_B(x_i))\,|\,x_i\in X\}$ are two IFSs, then a similarity measure can be defined as

$$S_R(A,B) = 1 - \frac{1}{3n}\sum_{i=1}^{n}(|\mu_A^2(x_i) - \mu_B^2(x_i)| + |\upsilon_A^2(x_i) - \upsilon_B^2(x_i)| + |m_A^2(x_i) - m_B^2(x_i)|) \quad (1)$$

where $m_A(x_i) = \frac{\mu_A(x_i) + 1 - \upsilon_A(x_i)}{2}$ and $m_B(x_i) = \frac{\mu_B(x_i) + 1 - \upsilon_B(x_i)}{2}$.

Theorem 1. The measure $S_N(A,B)$ is a similarity measure of IFSs.

Proof. To illustrate $S_N(A,B)$ being a similarity measure of IFSs, we only need to prove it satisfies the properties in Definition 4.

(i) Obviously, for every x_i, we have

$$0 \le |\mu_A^2(x_i) - \mu_B^2(x_i)| \le 1,\ 0 \le |\upsilon_A^2(x_i) - \upsilon_B^2(x_i)| \le 1 \text{ and } 0 \le |m_A^2(x_i) - m_B^2(x_i)| \le 1$$

then $0 \le S_R(A,B) \le 1$.

(ii) If $A = B$, we have $\mu_A(x_i) = \mu_B(x_i), \upsilon_A(x_i) = \upsilon_B(x_i), \forall x_i \in X$. Then $S_R(A,B) = 1$.

(iii) $S_R(B,A) = 1 - \frac{1}{3n}\sum_{i=1}^{n}(|\mu_B^2(x_i) - \mu_A^2(x_i)| + |\upsilon_B^2(x_i) - \upsilon_A^2(x_i)| + |m_B^2(x_i) - m_A^2(x_i)|) = S_R(A,B)$

(iv) If $A \subseteq B \subseteq C$, we have for $\forall x_i \in X$, $\mu_A(x_i) \le \mu_B(x_i) \le \mu_C(x_i), \upsilon_A(x_i) \ge \upsilon_B(x_i) \ge \upsilon_C(x_i)$ .Then it is obviously that

$$|\mu_A^2(x_i) - \mu_B^2(x_i)| \le |\mu_A^2(x_i) - \mu_C^2(x_i)|, |\mu_B^2(x_i) - \mu_C^2(x_i)| \le |\mu_A^2(x_i) - \mu_C^2(x_i)|$$

$$|\upsilon_A^2(x_i) - \upsilon_B^2(x_i)| \le |\upsilon_A^2(x_i) - \upsilon_C^2(x_i)|,\ |\upsilon_B^2(x_i) - \upsilon_C^2(x_i)| \le |\upsilon_A^2(x_i) - \upsilon_C^2(x_i)|$$

and

$$\mu_A(x_i) - \upsilon_A(x_i) \le \mu_B(x_i) - \upsilon_B(x_i) \le \mu_C(x_i) - \upsilon_C(x_i) \text{ i.e. } S_A(x_i) \le S_B(x_i) \le S_C(x_i)$$

Then

$$|S_B(x_i) - S_A(x_i)| \le |S_C(x_i) - S_A(x_i)|, S_A(x_i) + S_B(x_i) \le S_A(x_i) + S_C(x_i)$$

and

$$|S_C(x_i) - S_B(x_i)| \le |S_C(x_i) - S_A(x_i)|, S_B(x_i) + S_C(x_i) \le S_A(x_i) + S_C(x_i)$$

Consequently

$$\begin{aligned} &|m_A^2(x_i) - m_B^2(x_i)| = |m_A(x_i) - m_B(x_i)| \times (m_A(x_i) + m_B(x_i)) \\ &= \frac{1}{4}|S_A(x_i) - S_B(x_i)|(2 + S_A(x_i) + S_B(x_i)) \\ &\le \frac{1}{4}|S_A(x_i) - S_C(x_i)|(2 + S_A(x_i) + S_C(x_i)) \\ &= |m_A^2(x_i) - m_C^2(x_i)| \end{aligned}$$

and

$$
\begin{aligned}
&| m_B^2(x_i) - m_C^2(x_i) | \\
&= \frac{1}{4} | S_B(x_i) - S_C(x_i) | (2 + S_B(x_i) + S_C(x_i)) \\
&\leq \frac{1}{4} | S_A(x_i) - S_C(x_i) | (2 + S_A(x_i) + S_C(x_i)) \\
&= | m_A^2(x_i) - m_C^2(x_i) |
\end{aligned}
$$

And we can prove $S_R(A,B) \geq S_R(A,C)$ and $S_R(B,C) \geq S_R(A,C)$.

(v) If A is a crisp set, we have $\mu_A(x_i) = 0$ or 1, for $\forall x_i \in X$,

(a) If $\forall x_i \in X$, $\mu_A(x_i) = 0$, we have $\upsilon_A(x_i) = 1$, $m_A(x_i) = \dfrac{\mu_A(x_i) + 1 - \upsilon_A(x_i)}{2} = 0$, $\mu_{A^C}(x_i) = 1$, $\upsilon_{A^C}(x_i) = 0$ and $m_{A^C}(x_i) = \dfrac{\mu_{A^C}(x_i) + 1 - \upsilon_{A^C}(x_i)}{2} = 1$

Then

$$
\begin{aligned}
S_R(A, A^C) &= 1 - \frac{1}{3n} \sum_{i=1}^{n} (| \mu_A^2(x_i) - \mu_{A^C}^2(x_i) | + | \upsilon_A^2(x_i) - \upsilon_{A^C}^2(x_i) | + | m_A^2(x_i) - m_{A^C}^2(x_i) |) \\
&= 1 - \frac{1}{3n} \sum_{i=1}^{n} (| 0 - 1 | + | 1 - 0 | + | 0 - 1 |) = 0
\end{aligned}
$$

(b) for $\forall x_i \in X$, $\mu_A(x_i) = 1$, similarly to the proof of case (a), we can prove $S_R(A, A^C) = 0$.□

If we consider the important degree of x_i, a weighted similarity measure between IFS A and B is proposed as follows:

$$
S_{WR}(A,B) = 1 - \frac{1}{3} \sum_{i=1}^{n} w_i (| \mu_A^2(x_i) - \mu_B^2(x_i) | + | \upsilon_A^2(x_i) - \upsilon_B^2(x_i) | + | m_A^2(x_i) - m_B^2(x_i) |)
$$

where $w_i \in [0,1]$ $(i = 1, 2, ..., n)$ is the important degree of x_i, and $\sum_{i=1}^{n} w_i = 1$.

If we set $w_i = 1/n$ $(i = 1, 2, ..., n)$, then $S_{WR}(A,B) = S_R(A,B)$. Similar to the proof process of $S_R(A,B)$, we can easily prove that the weighted similarity measure $S_{WR}(A,B)$ also satisfies the conditions in Definition 4,

Remark 2. To demonstrate the reasonability of the new proposed similarity measure, the similarity measures are calculated with respect to the above mentioned IFSs:

$S_R(A_0, B_0) = 0.8333$, $S_R(A_1, B_1) = 0.9533, S_R(C_1, D_1) = 0.9200$, $S_R(A_2, B_2) = 0.9533, S_R(A_2, C_2) = 0.9492$.

The result seems to be reasonable and the proposed similarity measures can have a stronger discrimination among them.

In the follows, we will give two examples in pattern recognition and medical diagnosis to demonstrate the effectiveness and practicability of the proposed similarity measure.

Example 1 We consider the pattern recognition problem discussed in (Li and Chen [20]; Ye [27]). There are three patterns C_1, C_2 and C_3, which are represented by the following IFSs in the given finite universe $X = \{x_1, x_2, x_3\}$, respectively:

$$C_1 = \{(x_1, 1.0, 0.0), (x_2, 0.8, 0.0), (x_3, 0.7, 0.1)\}$$
$$C_2 = \{(x_1, 0.8, 0.1), (x_2, 1.0, 0.0), (x_3, 0.9, 0.0)\}$$
$$C_3 = \{(x_1, 0.6, 0.2), (x_2, 0.8, 0.0), (x_3, 1.0, 0.0)\}$$

Given an unknown pattern Q, which is represented by the IFS:

$$Q = \{(x_1, 0.5, 0.3), (x_2, 0.6, 0.2), (x_3, 0.8, 0.1)\}.$$

The task is to classify the pattern Q in one of the class C_1, C_2 and C_3. According to the recognition principle of maximum degree of similarity between IFSs, the process of assigning the pattern Q to C_k is described by

$$k = \arg\max_{1 \le i \le 3} \{S_R(C_i, Q)\} \qquad .$$

(2)

By Eq.(1), we can compute the similarity measure between $C_i (i = 1, 2, 3)$ with Q :

$S_R(C_1, Q) = 0.7386$, $S_R(C_2, Q) = 0.7353$ and $S_R(C_3, Q) = 0.8247$

Then we can observe that the pattern Q should be classified in C_3 according to the recognition rule given by Eq. (2). This result is in agreement with the one obtained in (Li and Chen [20]; Ye [27]).

Example 2 We consider the medical diagnosis problem discussed in (Vlachos and Sergiadis [34]). Let us consider a set of diagnoses Q= { Q_1 (Viral fever), Q_2 (Malaria), Q_3 (Typhoid), Q_4 (Stomach problem), Q_5 (Chest problem)}, and a set of symptoms S ={ s_1 (Temperature), s_2 (Headache), s_3 (Stomach pain), s_4 (Cough), s_5 (Chest pain)}.

Each diagnosis $Q_i (i = 1, 2, 3, 4, 5)$ can be represented by the following IFSs, respectively:

$$Q_1 = \{(s_1, 0.4, 0.0), (s_2, 0.3, 0.5), (s_3, 0.1, 0.7), (s_4, 0.4, 0.3), (s_5, 0.1, 0.7)\}$$
$$Q_2 = \{(s_1, 0.7, 0.0), (s_2, 0.2, 0.6), (s_3, 0.0, 0.9), (s_4, 0.7, 0.0), (s_5, 0.1, 0.8)\}$$
$$Q_3 = \{(s_1, 0.3, 0.3), (s_2, 0.6, 0.1), (s_3, 0.2, 0.7), (s_4, 0.2, 0.6), (s_5, 0.1, 0.9)\}$$
$$Q_4 = \{(s_1, 0.1, 0.7), (s_2, 0.2, 0.4), (s_3, 0.8, 0.0), (s_4, 0.2, 0.7), (s_5, 0.2, 0.7)\}$$
$$Q_5 = \{(s_1, 0.1, 0.8), (s_2, 0.0, 0.8), (s_3, 0.2, 0.8), (s_4, 0.2, 0.8), (s_5, 0.8, 0.1)\}$$

Suppose a patient P named Bob, with respect to all the symptoms, can be represented by the following IFS:

$$P = \{(s_1, 0.0, 0.8), (s_2, 0.4, 0.4), (s_3, 0.6, 0.1), (s_4, 0.1, 0.7), (s_5, 0.1, 0.8)\}$$

Our aim is to determine the patient P belong to which diagnosis of $Q_i\,(i=1,2,3,4,5)$. Because the medical diagnosis problem is actually a pattern recognition problem, then we can use the recognition rule given as follows:

If $k=\arg\max\limits_{1\le i\le 5}\{S_R(Q_i,P)\}$, then we assign the patient P to the diagnosis Q_k.

By the Eq. (2), we can obtain the following results:

$S_R(Q_1,P)=0.7493, S_R(Q_2,P)=0.6272, S_R(Q_3,P)=0.7912, S_R(Q_4,P)=0.9125$, and $S_R(Q_5,P)=0.7058$

Then, we can assign the patient P to the diagnosis Q_4 (Stomach problem), and the result is in agreement with the one obtained in (Vlachos and Sergiadis [34]).

4. Intuitionistic Fuzzy MADM Problem

In this section, we will propose a new decision making method for the MADM problem in which the attribute values are expressed by intuitionistic fuzzy numbers based on the proposed similarity measures of IFSs. Firstly, we will introduce the MADM model, and then give the specific calculation steps.

For a MADM problem, let $A=\{A_1,A_2,\cdots,A_m\}$ is a set of m alternatives, $O=\{o_1,o_2,\cdots,o_n\}$ is a set of n attributes. Suppose that there exists an alternative set consisting of n parallel alternatives from which the most desirable alternative is to be selected. Ratings of alternatives $A_i\in A$ on attributes $o_j\in O$ are expressed with the intuitionistic fuzzy number $\tilde{a}_{ij}=(\mu_{ij},\upsilon_{ij})$, respectively, where μ_{ij} and υ_{ij} are the membership (satisfactory) and non-membership (non-satisfactory) degrees of the alternative $A_i\in A$ on the attribute $o_j\in O$, with respect to the fuzzy concept "excellence" given by the decision maker, so that they satisfy the conditions: $0\le\mu_{ij}\le 1$, $0\le\upsilon_{ij}\le 1$ and $0\le\mu_{ij}+\upsilon_{ij}\le 1$ ($i=1,2,\cdots,n$; $j=1,2,\cdots,m$).

Thus, a MADM problem can be expressed with the decision matrix $\boldsymbol{D}=(\tilde{a}_{ij})_{m\times n}$:

$$\boldsymbol{D}=(\tilde{a}_{ij})_{m\times n}=\begin{array}{c} \\ A_1\\ A_2\\ \vdots\\ A_m\end{array}\begin{array}{c}\begin{array}{cccc} o_1 & o_2 & \cdots & o_n\end{array}\\ \begin{pmatrix}(\mu_{11},\upsilon_{11}) & (\mu_{12},\upsilon_{12}) & \cdots & (\mu_{1n},\upsilon_{1n})\\ (\mu_{21},\upsilon_{21}) & (\mu_{22},\upsilon_{22}) & \cdots & (\mu_{2n},\upsilon_{2n})\\ \vdots & \vdots & \vdots & \vdots\\ (\mu_{m1},\upsilon_{m1}) & (\mu_{m2},\upsilon_{m2}) & \cdots & (\mu_{mn},\upsilon_{mn})\end{pmatrix}\end{array}.$$

Let $\boldsymbol{\omega}=(\omega_1,\omega_2,\cdots,\omega_m)^{\mathrm{T}}$ be the weight vector of all attributes, where $0\le\omega_j\le 1$ ($j=1,2,\cdots,m$) is weight of each attribute $o_j\in O$, and $\sum_{j=1}^{m}\omega_j=1$. The attribute weights information is usually unknown or partially known due to the insufficient knowledge or limitation of time of decision makers in the decision making process. Therefore, the determination of attribute weights is an important issue in MADM problems. In this paper, we will put forward two methods to determine the attribute weights for the above-mentioned two cases, respectively.

(1) MADM problem with unkown attribute weights information

It is easily shown that $d(A,B)=1-S_R(A,B)$ is a distance measure of IFSs $A=\{(x_i,\mu_A(x_i),\upsilon_A(x_i))\,|\,x_i\in X\}$ and $B=\{(x_i,\mu_B(x_i),\upsilon_B(x_i))\,|\,x_i\in X\}$. Then when the information about the attribute weights is completely unknown, we can use the maximizing deviation method (Wang, 1998) to derive the weights of attributes with the following formula:

$$w_j=\frac{\sum_{i=1}^{m}\sum_{k=1}^{m}d(\tilde{a}_{ij},\tilde{a}_{kj})}{\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m}d(\tilde{a}_{ij},\tilde{a}_{kj})},\quad j=1,2,...,n \tag{3}$$

where $d(\tilde{a}_{ij},\tilde{a}_{kj})$ is the distance between $\tilde{a}_{ij}$ and $\tilde{a}_{kj}$ defined by $d(\tilde{a}_{ij},\tilde{a}_{kj})=1-S_R(\tilde{a}_{ij},\tilde{a}_{kj})$.

(2) MADM problem with partially known attribute weights information

In real decision situations, due to the complexity and uncertainty of practical decision making problems and the inherent subjective nature of human thinking, the attribute weights information is usually partially known [36-40]. Generally, there will have more constraint conditions for weight vector $\boldsymbol{w}=(w_1,w_2,...,w_n)^T$. We denote $\boldsymbol{H}$ as the set of the partially known weight information, where H can be constructed by the following forms, for $i\neq j$:

Case (i): A weak ranking: $\{w_i\geq w_j\}$

Case (ii): A strict ranking: $\{w_i-w_j\geq\alpha_i\}(\alpha_i>0)$;

Case (iii): A ranking of differences: $\{w_i-w_j\geq w_k-w_l\}$, for $j\neq k\neq l$;

Case (iv): A ranking with multiples: $\{w_i\geq\alpha_i w_j\}$ $(0\leq\alpha_i\leq 1)$;

Case (v): An interval form: $\{\alpha_i\leq w_i\leq\alpha_i+\varepsilon_i\}$ $(0\leq\alpha_i\leq\alpha_i+\varepsilon_i\leq 1)$.

To determine the attribute weights for MADM problem with partially known attribute weights information under intuitionistic fuzzy environment, Xu [40] proposed an optimization model based on the Chen and Tan's score function [41]; Wu and Zhang [42], Wang and Wang [43] determined the attribute weights by establishing a programming model according to the minimum entropy principle. In this paper, we will use the new similarity measure to determine the attribute weights, and the method is similarly with Chen and Yang [36]. Then for the MADM problem $\tilde{\boldsymbol{D}}=(\tilde{a}_{ij})_{n\times m}$ with incomplete attribute weight information $\boldsymbol{w}\in\boldsymbol{H}$, the decision making process is given as follows:

To rank the alternatives according to the decision matrix $D=(\tilde{a}_{ij})_{m\times n}$, we propose a method to obtain the attribute weight vector by means of the proposed similarity measure of IFSs. Similarity measure describes the degree of similarity between two IFSs. The weighted similarity measure of alternative $A_i=(\tilde{a}_{i1},\tilde{a}_{i2},...,\tilde{a}_{in})$ with the intuitionistic fuzzy positive ideal solution $A^{+}=(\tilde{a}_1^{+},\tilde{a}_2^{+},...,\tilde{a}_n^{+})=((1,0),(1,0),...,(1,0))$ is defined as

$$S(A_i)=\sum_{j=1}^{n}w_jS_R(\tilde{a}_{ij},\tilde{a}_j^{+})=\sum_{j=1}^{n}w_j[2-\mu_{ij}^2+\upsilon_{ij}^2-\frac{1}{4}\times(\mu_{ij}+1-\upsilon_{ij})^2] \tag{4}$$

The larger the similarity measure between the alternative with positive ideal solution, the better of the alternative is.

Hence, we can utilize the principle of maximization of similarity measure to get the attribute weight vector by computing the following programming:

$$\begin{aligned} &\max\ S(A_i)=\sum_{j=1}^{n}w_jS_R(\tilde{a}_{ij},\tilde{a}_j^{+})\\ &s.t.\ \ \sum_{j=1}^{n}w_j=1\\ &\qquad \boldsymbol{w}\in H \end{aligned} \tag{5}$$

where $\boldsymbol{H}$ is the information set of attribute weights.

A reasonable attribute weights vector $\boldsymbol{w}=(w_1,w_2,...,w_n)$ should make all the similarities $\{S(A_i)\,|\,i=1,2,...,m\}$ as large as possible under the condition $\boldsymbol{w}\in\boldsymbol{H}$. And each alternative is equally likely important, hence we can establish the following optimization programming:

$$\begin{aligned} &\max\ S=\sum_{i=1}^{m}S(A_i)=\sum_{i=1}^{m}\sum_{j=1}^{n}w_jS_R(\tilde{a}_{ij},\tilde{a}_j^{+})\\ &s.t.\ \ \sum_{j=1}^{n}w_j=1\\ &\qquad \boldsymbol{w}\in\boldsymbol{H} \end{aligned}$$

(6)

By solving the Eq. (6), the optimal solution $\boldsymbol{w}^{*}=\arg\max S$ is chosen as the optimal attribute weights.

(3) The New MADM Method

In this subsection, we put forward the new MADM method based on the above-mentioned work and the concept of TOPSIS. The specific calculation steps are given as follows:

Step 1. Calculate the attribute weights according to section (1) and section (2);

Step 2. Determine the positive ideal solution (PIS) and negative ideal solution (NIS) of the intuitionistic fuzzy MADM problem.

The PIS is defined as $A^{+}=(\tilde{a}_1^{+},\tilde{a}_2^{+},\cdots,\tilde{a}_n^{+})$, where $\tilde{a}_j^{+}=(\mu_j^{+},\upsilon_j^{+})=(1,0)$ ($j=1,2,\cdots,n$).

The NIS is defined as $A^{+}=(\tilde{a}_1^{-},\tilde{a}_2^{-},\cdots,\tilde{a}_n^{-})$, where $\tilde{a}_j^{-}=(\mu_j^{-},\upsilon_j^{-})=(0,1)$ ($j=1,2,\cdots,n$).

Step 3. According to the weighted similarity measure defined in Eq. (4), the similarity measures between alternative A_i with PIS and NIS are calculated respectively as follows:

$$S_i^{+}=\sum_{j=1}^{n}w_j S_R(\tilde{a}_{ij},\tilde{a}_j^{+})=\sum_{j=1}^{n}w_j[2-\mu_{ij}^2+\upsilon_{ij}^2-\frac{1}{4}\times(\mu_{ij}+1-\upsilon_{ij})^2] \tag{7}$$

$$S_i^{-}=\sum_{j=1}^{n}w_j S_R(\tilde{a}_{ij},\tilde{a}_j^{-})=\sum_{j=1}^{n}w_j[1+\mu_{ij}^2-\upsilon_{ij}^2+\frac{1}{4}\times(\mu_{ij}+1-\upsilon_{ij})^2] \tag{8}$$

Step 4. Calculate the relative closeness coefficient of each alternative.

The closeness coefficient C_i represents the degree of similarity of each alternative to the PIS and NIS simultaneously. The closeness coefficient of each alternative is calculated as:

$$C_i=\frac{S_i^{+}}{S_i^{+}+S_i^{-}}\ (i=1,2,...,m) \tag{9}$$

Step 5. Rank the alternatives according to the closeness coefficient (C_i) in decreasing order. The best alternative is closest to the PIS and farthest from the NIS.

Remark 3. By Eq.(7) and Eq.(8), it is easy to prove $S_i^{+}+S_i^{-}=1$, the closeness coefficient C_i is equal to S_i^{+}. Thus the above steps 3-5, can be briefly substituted by the following step:

Step 3'. Calculate the similarity measures S_i^{+} between alternative A_i with PIS, and rank the alternatives according to S_i^{+} in decreasing order.

5. Discussion

In order to illustrate the effectiveness and practicability of the proposed MADM method, two examples are given as follows:

Example 3 (The attribute weights are complete unknown) Suppose that a company wants to invest a sum of money in the best options which adopted from (Herrera and Herrera-Viedma [44]; Ye [45]). There are four parallel alternatives to be selected: A_1 (a car company), A_2 (a food company), A_3 (a computer company), and A_4 (an arms company). The evaluation attributes are o_1 (the risk analysis), o_2 (the growth analysis), and o_3 (the environmental impact analysis). Using statistical methods, the membership degree μ_{ij} (i.e. μ_{ij} means the satisfactory degree) and non-membership degree υ_{ij} (i.e. υ_{ij} means the non-satisfactory degree) for the alternative A_i satisfies the attribute o_j can be obtained, respectively. The intuitionistic fuzzy decision matrix provided by relevance experts is shown in Table 1.

Table 1. Intuitionistic fuzzy decision matrix

Alternative	Evaluation attribute		
	o_1	o_2	o_3
A_1	(0.45,0.35)	(0.50,0.30)	(0.20,0.55)
A_2	(0.65,0.25)	(0.65,0.25)	(0.55,0.15)
A_3	(0.45,0.35)	(0.55,0.35)	(0.55,0.20)
A_4	(0.75,0.15)	(0.65,0.20)	(0.35,0.15)

The calculation steps of the proposed method are given as follows:

Step1. According to Eq. (3), the attribute weights vector is obtained as

$$\boldsymbol{w} = (w_1, w_2, w_3)^T = \left(0.3827, 0.2060, 0.4113\right)^T$$

Step 2. The PIS (A^+) is defined as:

$$A^+ = (\tilde{a}_1^+, \tilde{a}_2^+, \tilde{a}_3^+) = ((1,0),(1,0),(1,0))$$

Step 3. According to Eq. (7), the similarity measures of each alternative from PIS are calculated as

$$S_1^+ = 0.3963, S_2^+ = 0.6057, S_3^+ = 0.5177 \text{ , and } S_4^+ = 0.6093 .$$

Therefore, the ranking order of all alternatives is $A_4 \succ A_2 \succ A_3 \succ A_1$, and A_4 is the desirable alternative. This result is different with the one obtained in (Ye [45]), which the ranking order is $A_2 \succ A_4 \succ A_3 \succ A_1$. The reason is that Ye's [45] cosine similarity measure is only to consider the information of membership degree and non-membership degree, but not to consider the middle point information. Thus Ye's cosine similarity measure has some drawbacks which was reviewed in Section 2.

Example 4 (The attribute weights are partially known). The example is adopted from Li [46], which considers an air-condition system selection problem. Suppose there are three air-condition systems: A_i $(i = 1,2,3)$ are to be selected. The evaluation attributes are o_1 (economical), o_2 (function), and o_3 (being operative). Using statistical methods, we can obtain the membership degree μ_{ij} and non-membership degree υ_{ij} for the alternative A_i satisfying the attribute o_j respectively. The IF decision matrix provided by relevance experts is shown in Table 2.

Table 2. Intuitionistic fuzzy decision matrix

Air-condition system	Evaluation attribute		
	o_1	o_2	o_3
A_1	(0.75,0.1)	(0.6,0.25)	(0.8,0.2)
A_2	(0.8,0.15)	(0.68,0.2)	(0.45,0.5)
A_3	(0.4,0.45)	(0.75,0.05)	(0.6,0.3)

Assume the attribute weights are partially known, and the weights satisfies the set

$$\boldsymbol{H} = \{0.25 \le w_1 \le 0.75, 0.35 \le w_2 \le 0.60, 0.30 \le w_3 \le 0.35\}.$$

Then the calculation steps for the proposed decision making method is:

Step1. According to the Eq. (6), we can establish the following programming model:

$$\max S = 1.9048w_1 + 2.0019w_2 + 1.7035w_3$$

$$s.t. \begin{cases} 0.25 \le w_1 \le 0.75 \\ 0.35 \le w_2 \le 0.60 \\ 0.30 \le w_3 \le 0.35 \\ w_1 + w_2 + w_3 = 1 \end{cases}$$

We use Matlab software to solve this model, and get the optimum attribute weight vector $\boldsymbol{w} = \left(0.25, 0.45, 0.30\right)^T$.

Step 2. The PIS (A^+) is defined as:

$$A^+ = (\tilde{a}_1^+, \tilde{a}_2^+, \tilde{a}_3^+) = ((1,0),(1,0),(1,0))$$

Step 3. According to Eq. (3), the weighted similarity measures of each alternative from PIS are calculated as

$$S_1^+ = 0.6731, S_2^+ = 0.6048, S_3^+ = 0.6102$$

Step 4. Based on values of S_i^+ ($i = 1,2,3$), the ranking order of the alternatives is $A_1 \succ A_3 \succ A_2$,and A_1 is the best desirable supplier, which is in agreement with the one obtained in (Li [46]).

Remark 4. Intuitionistic fuzzy sets are suitable to model the vague information occurred in many MADM problems. In this paper, we have proposed a new similarity measure of IFSs. The new proposed similarity measures can overcome the counter-intuitive cases mentioned in the existing similarity measure's articles. This paper has given the application of the proposed similarity measure in pattern recognition, and medical diagnosis. The analysis result proved that the decision making method based on the proposed similarity measure is effective and feasible.

6. Conclusions

Similarity measure is an important tool to measure the degree of resemblance between two intuitionistic fuzzy sets. In this paper, in order to overcome the counter-intuitive in some cases, a new similarity measure of intuitionistic fuzzy sets is constructed and successively applied in pattern recognition and medical diagnosis and decision making problem. The proposed similarity measure can also be applied to other areas, such as image processing, microelectronic fault analysis. Furthermore, we put forward a new decision making method for the MADM problem in which the attribute values are expressed by IFNs.

For the case of attribute weights complete unknown, we develop a weights determined method according to the information theory, and for the case of attribute weights partially known, we establish a optimization model using the proposed weighted similarity measure. Then combining with the concept of TOPSIS, we give the specific calculation steps of the new proposed decision making method. Based on the proposed intuitionistic similarity measure, a new attribute weights determination method is put forward, and then we use it to the multi-attribute decision making problem. Two numerical examples are used to illustrate the feasibility and practicability of the proposed MADM method. As a prospect, the MADM method proposed in this paper could be applied to other MADM problems, such as the evaluation project investment risk, site selection and credit evaluation.

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References

1. L.A. Zadeh, Fuzzy sets, 1965, Information & Control, vol. 8, no. 3, pp. 338-356.
2. K. Atanassov, Intuitionistic fuzzy sets, 1986, Fuzzy Sets Systems, vol. 20, no.1, pp. 87-96.

3. C. Tan, W. Yi, X. Chen, Generalized intuitionistic fuzzy geometric aggregation operators and their application to multi-criteria decision making, 2015, Journal of the Operational Research Society, vol. 66, no. 11, pp. 1919-1938.
4. H. Nguyen, A new knowledge-based measure for intuitionistic fuzzy sets and its application in multiple attribute group decision making, 2015, Expert Systems with Applications, vol. 42, no. 22, pp. 8766-8774.
5. D.F. Li, The GOWA operator based approach to multiattribute decision making using intuitionistic fuzzy sets, 2011, Mathematical and Computer Modelling, vol. 53, no. 5-6, pp.1182-1196.
6. Z.S. Xu, Approaches to multiple attribute group decision making based on intuitionistic fuzzy power aggregation operators, 2011, Knowledge-Based Systems, vol. 24, no. 6, pp.749-760.
7. M.M. Xia, Z.S. Xu, Entropy/cross entropy-based group decision making under intuitionistic fuzzy environment, 2012, Information Fusion, vol. 13, no.1, pp.31-47.
8. G.W. Wei, Gray relational analysis method for intuitionistic fuzzy multiple attribute decision making, 2011, Expert Systems with Applications, vol. 38, no. 9, pp.11671-11677.
9. Y.D. He, H.Y. Chen, L.G. Zhou, et al., Generalized intuitionistic fuzzy geometric interaction operators and their application to decision making, 2014, Expert Systems with Applications, vol. 41, no. 5, pp. 2484-2495.
10. Z. Pei, L. Zheng, A novel approach to multi-attribute decision making based on intuitionistic fuzzy sets, 2012, Expert Systems with Applications, vol. 39, no. 3, pp. 2560-2566.
11. T. Chaira, Accurate segmentation of leukocyte in blood cell images using Atanassov's intuitionistic fuzzy and interval Type II fuzzy set theory, 2014, Micron, vol. 61, pp. 1-8.
12. V. Khatibi, G.A. Montazer, Intuitionistic fuzzy set vs. fuzzy set application in medical pattern recognition, 2009, Artificial Intelligence in Medicine, vol. 47, no.1, pp. 43-52.
13. F.E. Boran, D. Akay, A biparametric similarity measure on intuitionistic fuzzy sets with applications to pattern recognition, 2014, Information Sciences, vol. 255, no. 28, pp. 45-57.
14. C.M. Hwang, M.S. Yang, W.L. Hung, M.G. Lee, A similarity measure of intuitionistic fuzzy sets based on the Sugeno integral with its application to pattern recognition, 2012, Information Sciences, vol. 189, no. 6, pp. 93-109.

15. G.A. Papakostas, A.G. Hatzimichailidis, V.G. Kaburlasos, Distance and similarity measures between intuitionistic fuzzy sets: A comparative analysis from a pattern recognition point of view, 2013, Pattern Recognition Letters, vol.34, no. 14, pp. 1609-1622.
16. T. Chaira, A.K. Ray, A new measure using intuitionistic fuzzy set theory and its application to edge detection, 2008, Applied Soft Computing, vol. 8, no. 2, pp. 919-927.
17. T. Chaira, A rank ordered filter for medical image edge enhancement and detection using intuitionistic fuzzy set, 2012, Applied Soft Computing, vol. 12, no. 4, pp. 1259-1266.
18. Y. Li, D. Olson, Q. Zheng, Similarity measures between intuitionistic fuzzy (vague) sets: a comparative analysis, 2007, Pattern Recognition Letters, vol. 28, no.2, pp. 278-285.
19. L. Baccour, A.M. Alimi, R.I. John, Similarity measures for intuitionistic fuzzy sets: State of the art, 2013, Journal of Intelligent & Fuzzy Systems, vol. 24, no. 1, pp. 37-49.
20. D.F. Li, C.T. Chen, New similarity measures of intuitionistic fuzzy sets and application to pattern recognition, 2002, Pattern Recognition Letters, vol. 23, vol. s1-3, pp.221-225.
21. Z. Liang, P. Shi, Similarity measures on intuitionistic fuzzy sets, 2003, Pattern Recognition Letters, vol. 24, pp. 2687-2693.
22. H.B. Mitchell, On the Dengfeng-Chuntian similarity measure and its application to pattern recognition, 2003, Pattern Recognition Letters, vol. 24, no.16, pp.3101-3104.
23. E. Szmidt, J. Kacprzyk, Distances between intuitionistic fuzzy sets, 2000, Fuzzy Sets and Systems, vol. 114, no. 3, pp. 505-518.
24. W.L. Hung, M.S. Yang, Similarity measures of intuitionistic fuzzy sets based on hausdorff distance, 2004, Pattern Recognition Letters, vol. 25, no. 14, pp.1603-1611.
25. W.L. Hung, M.S. Yang, Similarity measures of intuitionistic fuzzy sets based on Lp metric, 2007, International Journal of Approximate Reasoning, vol. 46, no. 1, pp. 120-136.
26. Z.S. Xu, J. Chen, An overview of distance and similarity measures of intuitionistic fuzzy sets, 2008, International Journal of Uncertainty, Fuzziness and Knowlege-Based Systems, vol. 16, no. 4, pp. 529-555.
27. J. Ye, Cosine similarity measures for intuitionistic fuzzy sets and their applications, 2011, Mathematical and Computer Modelling, vol. 53, no. 1-2, pp. 91-97.
28. C.M. Hwang, M.S. Yang, New construction for similarity measures between intuitionistic fuzzy sets based on lower, upper and middle fuzzy sets, 2013, International Journal of Fuzzy Systems, vol. 15, no. 3, pp. 371-378.

29. J.Q. Wang, K.J. Li, H.Y. Zhang, Interval-valued intuitionistic fuzzy multi-criteria decision-making approach based on prospect score function, 2012, Knowledge-Based Systems, vol. 27, no. 3, pp. 119-125
30. K.T. Atanassov, New operations defined over the intuitionistic fuzzy sets, 1994, Fuzzy Sets & Systems, vol. 61, no. 2, pp. 137-142.
31. S.M. Chen, Measures of similarity between vague sets, 1995, Fuzzy Sets and Systems, vol.74, no.2, pp.217-223.
32. D.H. Hong, C. Kim, A note on similarity measures between vague sets and between elements, 1999, Information Science, vol.115, no.1, pp. 83-96.
33. Y.H. Li, Z.X. Chi, D.Q. Yan, Similarity measures between vague sets and vague entropy, 2002, Computer Science, vol. 29, no. 12, pp.129-132.
34. I.K. Vlachos, G.D. Sergiadis, Intuitionistic fuzzy information-application to pattern recognition, 2007, Pattern Recognition Letters, vol. 28, no.2, pp.197-206.
35. Y.M. Wang, Using the method of maximizing deviations to make decision for multi-indices, 1998, Journal of System Engineering and Electronic, vol. 8, no. 3, pp. 24-26.
36. Z.P. Chen, W. Yang, A new multiple attribute group decision making method in intuitionistic fuzzy setting, 2011, Applied Mathematical Modelling, vol. 35, no. 9, pp. 4424-4437.
37. S.H. Kim, S.H. Choi, K. Kim, An interactive procedure for multiple attribute group decision making with incomplete information: Range-based approach,1999, European Journal of Operational Research, vol. 118, no. 1, pp.139-152.
38. S.H. Kim, B.S. Ahn, Interactive group decision making procedure under incomplete information, 1999, European Journal of Operational Research, vol.116, no. 3, pp. 498-507.
39. K.S. Park, Mathematical programming models for characterizing dominance and potential optimality when multicriteria alternative values and weights are simultaneously incomplete, 2004, IEEE Transactions on Systems, Man, and Cybernetics-Part A, Systems and Humans, vol.34, no. 5, pp. 601-614.
40. Z.S. Xu, Multi-person multi-attribute decision making models under intuitionistic fuzzy environment, 2007, Fuzzy Optimization Decision Making, vol. 6, no. 3, pp. 221-236.
41. S.M. Chen, J.M. Tan, Handing Multicriteria fuzzy decision-making problems based on vague set theory, 1994, Fuzzy Sets and Systems, vol. 67, no. 2, pp. 163-172.

42. J.Z. Wu, Q. Zhang, Multicriteria decision making method based on intuitionistic fuzzy weighted entropy, 2011, Expert Systems with Applications, vol.38, no.1, pp. 916-922.
43. J.Q. Wang and P. Wang, Intuitionistic linguistic fuzzy multi-criteria decision-making method based on intuitionistic fuzzy entropy, 2012, Control and Decision, vol. 27, no.11, pp.1694-1698.
44. F. Herrera, E.H. Viedma, Linguistic decision analysis: steps for solving decision problems under linguistic information, 2000, Fuzzy Sets and Systems, vol. 115, no. 1, pp.67-82.
45. J. Ye, Fuzzy decision-making method based on the weighted correlation coefficient under intuitionistic fuzzy environment, 2010, European Journal of Operational Research, vol. 205, no. 1, pp. 202-204.
46. D.F. Li, Multiattribute decision making models and methods using intuitionistic fuzzy sets, 2005, Journal of Computer and System Sciences, vol.70, no.1, pp.73-85.