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The Results of Computing the Special Determinant

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Abstract. In this paper the special determinants have been computed, which are the determinants of order 3n, and consist of 0,1,2 order differentials. The detailed is that: elements of roads from 1 to n in the determinants are the 0 order differentials of $x_i^j (i, j = 0,1,2...n)$.The elements of roads from n+1 to 2n in the determinants are the 1 order differentials of $x_i^j (i, j = 0,1,2...n)$. The elements of roads from 2n+1 to 3n in the determinants are the 2 order differentials of $x_i^j (i, j = 0,1,2...n)$.In the corresponding confections matrix of the equations of 3 order .the special determinants are used to determinant the uniqueness of the roots for the equations of 3 order .Without the computations of the special determinants, the solution for the equations of 3n order will become very difficult.

Key words : Interpolation approach, Function, Determinants

1 Introduction

The computation of special determinants has been used to simplify the addition for sever fractions in papers [1-8]. . They are the computation of high order determinants and their elements are consist of 0,1,2 order differentials whose elements of roads from 1 to n+1 in the determinants are the 0 order differentials of $x_i (i = 0,1,2,...,n.)$ .and the elements of roads from n+2 to 2n+2 in the determinants are the 1 order differentials of $\boldsymbol{x_i (i = 0,1,2...)}$.,and the elements of roads from 2n+3 to 3n+3 in the determinants are the

2 order differentials of $x_i(i = 0,1,2,...,n.)$ .Because of the high order ,to compute them are difficulty. But they have been reduced to the sum of several determinants ,which are lower-order. The several determinants of lower- order are easy to be computed .These results can be applied in interpolation method,Following is the results:

Lemma 1.1

$$D(3n+4)_{n+2,1} = (-1)^{3n+1} 2 \prod_{i=1}^{n} (x_i - x_0)^9 D(3n+1)_{n+1.1} \qquad (1.1)$$

Where $\boldsymbol{D(3n+4)_{n+2,1}}$ is the cofactor of following determinants

$$\boldsymbol{D(3n+4)} = \begin{vmatrix} 0 & 1 & x_0 & x_0^2 & x_0^3 & \cdots & x_0^{3n+2} \\ 1 & 1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^{3n+2} \\ & & & & & \cdots & \\ 1 & 1 & x_n & x_n^2 & x_n^3 & \cdots & x_n^{3n+2} \\ 1 & 1 & x & x^2 & x^3 & \cdots & x^{3n+2} \\ 0 & 0 & 1 & 2x_0 & 3x_0^2 & \cdots & (3n+2)x_0^{3n+1} \\ 0 & 0 & 1 & 2x_1 & 3x_1^2 & \cdots & (3n+2)x_1^{3n+1} \\ & & & & & \cdots & \\ 0 & 0 & 1 & 2x_n & 3x_n^2 & \cdots & (3n+2)x_n^{3n+1} \\ 0 & 0 & 0 & \frac{2!}{0!} & \frac{3!}{1!}x_0 & \cdots & \frac{(3n+2)!}{3n!}x_0^{3n} \\ 0 & 0 & 0 & \frac{2!}{0!} & \frac{3!}{1!}x_1 & \cdots & \frac{(3n+2)!}{3n!}x_1^{3n} \\ & & & & & \cdots & \\ 0 & 0 & 0 & \frac{2!}{0!} & \frac{3!}{1!}x_n & \cdots & \frac{(3n+2)!}{3n!}x_n^{3n} \end{vmatrix}. \qquad (1.2)$$

dispelled the entries of n+2-th row、1st column in determinant D(3n+4)and

$D(3n+1)_{n+1,1}$ is the cofactor deleted the entries of the n+1-th raw、1st column

in the following determinant D(3n+1):

$$D(3n+1)=\begin{vmatrix}
1 & 1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^{3n-1} \\
1 & 1 & x_2 & x_2^2 & x_2^3 & \cdots & x_2^{3n-1} \\
 & & & & & \cdots & \\
1 & 1 & x_n & x_n^2 & x_n^3 & \cdots & x_n^{3n-1} \\
1 & 1 & x & x^2 & x^3 & \cdots & x^{3n-1} \\
0 & 0 & 1 & 2x_1 & 3x_1^2 & \cdots & (3n-1)x_1^{3n-2} \\
0 & 0 & 1 & 2x_2 & 3x_2^2 & \cdots & (3n-1)x_2^{3n-2} \\
 & & & & & \cdots & \\
0 & 0 & 1 & 2x_n & 3x_n^2 & \cdots & (3n-1)x_n^{3n-2} \\
0 & 0 & 0 & \frac{2!}{0!} & \frac{3!}{1!}x_1 & \cdots & \frac{(3n-1)!}{(3n-3)!}x_1^{3n-3} \\
0 & 0 & 0 & \frac{2!}{0!} & \frac{3!}{1!}x_2 & \cdots & \frac{(3n-1)!}{(3n-3)!}x_2^{3n-3} \\
 & & & & & \cdots & \\
0 & 0 & 0 & \frac{2!}{0!} & \frac{3!}{1!}x_n & \cdots & \frac{(3n-1)!}{(3n-3)!}x_n^{3n-3}
\end{vmatrix}. \tag{1.3}$$

Proof. According to definition of algebra cofactor, $D(3n+4)_{n+2+1,1}$ is

$$D(3n+4)_{n+2+1}=(-1)^{n+2+1}\begin{vmatrix}
1 & x_0 & x_0^2 & x_0^3 & \cdots & x_0^{3n+2} \\
1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^{3n+2} \\
 & & & & \cdots & \\
 & & & & \cdots & \\
1 & x_n & x_n^2 & x_n^3 & \cdots & x_n^{3n+2} \\
0 & 1 & 2x_0 & 3x_0^2 & \cdots & (3n+2)x_0^{3n+1} \\
0 & 1 & 2x_1 & 3x_1^2 & \cdots & (3n+2)x_1^{3n+1} \\
 & & & & \cdots & \\
0 & 1 & 2x_n & 3x_n^2 & \cdots & (3n+2)x_n^{3n+1} \\
0 & 0 & \frac{2!}{0!} & \frac{3!}{1!}x_0 & \cdots & \frac{(3n+2)!}{3n!}x_0^{3n} \\
0 & 0 & \frac{2!}{0!} & \frac{3!}{1!}x_1 & \cdots & \frac{(3n+2)!}{3n!}x_1^{3n} \\
 & & & & \cdots & \\
0 & 0 & \frac{2!}{0!} & \frac{3!}{1!}x_n & \cdots & \frac{(3n+2)!}{3n!}x_n^{3n}
\end{vmatrix} \tag{1.4}$$

Beginning the last column in form (2.3) add $-x_1$ times the k-1-th column to the k-th column (k=3n+2,3n+1,3n,...2,1) .The result is:

$$D(3n+4)_{n+2,1} = (-1)^{n+3} A = (-1)^{n+1} A. \qquad (1.5)$$

Where:

$$A = \begin{vmatrix}
1 & 0 & 0 & 0 & \ldots & 0 \\
1 & x_1 - x_0 & x_1^2 - x_1 x_0 & x_1^3 - x_1^2 x_0 & \cdots & x_1^{3n+2} - x_1^{3n+1} x_0 \\
1 & x_2 - x_0 & x_2^2 - x_2 x_0 & x_2^3 - x_2^2 x_0 & \cdots & x_2^{3n+2} - x_2^{3n+1} x_0 \\
 & & & & \cdots & \\
 & & & & \cdots & \\
1 & x_{n-1} - x_0 & x_{n-1}^2 - x_{n-1} x_0 & x_{n-1}^3 - x_{n-1}^2 x_0 & \cdots & x_{n-1}^{3n+2} - x_{n-1}^{3n+1} x_0 \\
1 & x_n - x_1 & x_n^2 - x x_1 & x_n^3 - x_n x_0 & \cdots & x_n^{3n+2} - x_n^{3n+1} x_0 \\
0 & 1 & 2x_0 - x_0 & 3x_0^2 - 2x_0 x_0 & \cdots & (3n+2)x_0^{3n+1} - (3n+1)x_0^{3n} x_0 \\
0 & 1 & 2x_1 - x_1 & 3x_1^2 - 2x_1 x_1 & \cdots & (3n+2)x_1^{3n+1} - (3n+1)x_1^{3n} x_0 \\
0 & 1 & 2x_2 - x_1 & 3x_2^2 - 2x_2 x_1 & \cdots & (3n+2)x_2^{3n+1} - (3n+1)x_2^{3n} x_0 \\
 & & & & \cdots & \\
0 & 1 & 2x_n - x_1 & 3x_n^2 - 2x_n x_1 & \cdots & (3n+2)x_n^{3n+1} - (3n+1)x_n^{3n} x_0 \\
0 & 0 & \frac{2!}{0!} & \frac{3!}{1!}x_0 - \frac{2!}{0!}x_0 & \cdots & \frac{(3n+2)!}{3n!}x_0^{3n-1} - \frac{(3n+1)!}{(3n-1)!}x_0^{3n-2}x_0 \\
0 & 0 & \frac{2!}{0!} & \frac{3!}{1!}x_1 - \frac{2!}{0!}x_0 & \cdots & \frac{(3n+1)!}{3n!}x_1^{3n-1} - \frac{(3n+1)!}{(3n-1)!}x_1^{3n-2}x_0 \\
 & & & & \cdots & \\
 & & & & \cdots & \\
0 & 0 & \frac{2!}{0!} & \frac{3!}{1!}x_n - \frac{2!}{0!}x_0 & \cdots & \frac{(3n+1)!}{3n!}x_n^{3n-1} - \frac{3n!}{(3n-1)!}x_n^{3n-2}x_0
\end{vmatrix} \qquad (1.6)$$

Expand the determinant along rows 1 and then extract out the common factor $x_i - x_0$ from rows i in $D(3n+3)_{1,1}$, (i=1,2,...,n) , So above determinant becomes :

$$A = (-1)^{1+1} \prod_{i=1}^{n} (x_i - x_0) B \qquad (1.7)$$

Where B is following determinants :

$$
B=\begin{vmatrix}
1 & x_1 & x_1^2 & \cdots & x_1^{3n+1} \\
1 & x_2 & x_2^2 & \cdots & x_2^{3n+1} \\
1 & x_3 & x_3^2 & \cdots & x_3^{3n+1} \\
 & & & \cdots & \\
 & & & \cdots & \\
1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{3n+1} \\
1 & x_n & x_n^2 & \cdots & x_n^{3n+1} \\
1 & x_0 & x_0^2 & \cdots & x_0^{3n+1} \\
1 & 2x_1-x_0 & 3x_1^2-2x_1x_0 & \cdots & (3n+2)x_1^{3n+1}-(3n+1)x_1^{3n}x_0 \\
1 & 2x_2-x_0 & 3x_2^2-2x_2x_0 & \cdots & (3n+2)x_2^{3n+1}-(3n+1)x_2^{3n}x_0 \\
 & & & \cdots & \\
1 & 2x_n-x_0 & 3x_n^2-2x_nx_0 & \cdots & (3n+2)x_n^{3n+1}-(3n+1)x_n^{3n}x_0 \\
0 & \frac{2!}{0!} & \frac{3!}{0!}x_0-\frac{2!}{0!}x_0 & \cdots & \frac{(3n+2)!}{3n!}x_0^{3n}-\frac{(3n+1)!}{(3n-1)!}x_1^{3n-1}x_0 \\
0 & \frac{2!}{0!} & \frac{3!}{1!}x_1-\frac{2!}{0!}x_0 & \cdots & \frac{(3n+2)!}{3n!}x_1^{3n}-\frac{(3n+1)!}{(3n-1)!}x_1^{3n-1}x_0 \\
 & & & \cdots & \\
 & & & \cdots & \\
0 & \frac{2!}{0!} & \frac{3!}{1!}x_n-\frac{2!}{0!}x_0 & \cdots & \frac{(3n+2)!}{3n!}x_n^{3n}-\frac{(3n+1)!}{(3n-1)!}x_n^{3n-1}x_0
\end{vmatrix}
\qquad (1.8).
$$

In form (1.8), line $n+1+i$ $(i=1,2,...,n.)$ elements minus the line i $(i=1,2,...,n.)$ elements ,and exact out the common factors ,

determinants (1.8) became :

$$
B=\prod_{i=2}^{n}(x_i-x_1)C \qquad (1.9)
$$

$$C=2\begin{vmatrix}
1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^{3n+1} \\
 & & & & \cdots & \\
1 & x_{n-1} & x_{n-2}^2 & x_{n-3}^3 & \cdots & x_n^{3n+1} \\
1 & x_n & x_n^2 & x_n^3 & \cdots & x_n^{3n+1} \\
1 & x_0 & x_0^2 & x_0^3 & \cdots & x_0^{3n+1} \\
0 & 1 & 2x_1 & 3x_1^2 & \cdots & (3n+1)x_1^{3n} \\
0 & 1 & 2x_2 & 3x_2^2 & \cdots & (3n+1)x_2^{3n} \\
 & & & & \cdots & \\
0 & 1 & 2x_n & 3x_n^2 & \cdots & (3n+1)x_n^{3n} \\
0 & 1 & 2x_0 & 3x_0^2 & \cdots & 3nx_0^{3n-1} \\
0 & \frac{2!}{01} & \frac{3!}{1!}x_1-\frac{2!}{0!}x_0 & \frac{4!}{2!}x_1^2-\frac{3!}{1!}x_1x_0 & \cdots & \frac{(3n+1)!}{(3n-1)}x_1^{3n-2}-\frac{3n!}{(3n-2)!}x_1^{3n-3}x_0 \\
 & & & & \cdots & \\
0 & \frac{2!}{0!} & \frac{3!}{1!}x_n-\frac{2!}{0!}x_0 & \frac{4!}{2!}x_n^2-\frac{3!}{1!}x_nx_0 & \cdots & \frac{(3n+1)!}{(3n-1)!}x_n^{3n-2}-\frac{3n!}{(3n-2)!}x_n^{3n-1}x_0
\end{vmatrix}$$

(1.10)

The elements in rows 2n+2+i (i=1,2,...n) minus two times of the elements in n+2+i (i=1,2,...n) rows ,and then exact out the common factor $x_i - x_0$

$$C=2\prod_{i=1}^{n}(x_i-x_0)\begin{vmatrix}
1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^{3n} \\
 & & & & \cdots & \\
1 & x_{n-1} & x_{n-1}^2 & x_{n-1}^3 & \cdots & x_{n-1}^{3n} \\
1 & x_n & x_n^2 & x_n^3 & \cdots & x_n^{3n} \\
1 & x_0 & x_0^2 & x_0^3 & \cdots & x_0^{3n} \\
0 & 1 & 2x_1 & 3x_1^2 & \cdots & 3nx_1^{3n-1} \\
0 & 1 & 2x_2 & 3x_2^2 & \cdots & 3nx_2^{3n-1} \\
 & & & & \cdots & \\
0 & 1 & 2x_n & 3x_n^2 & \cdots & 3nx_n^{3n-1} \\
0 & 1 & 2x_0 & 3x_0 & \cdots & 3nx_1^{3n-2} \\
0 & 0 & \frac{2!}{0!} & \frac{3!}{1!}x_1 & \cdots & \frac{3n!}{(3n-2)!}x_1^{3n-2} \\
 & & & & \cdots & \\
0 & 0 & \frac{2!}{0!} & \frac{3!}{1!}x_n & \cdots & \frac{3n!}{(3n-2)!}x_n^{3n-1}
\end{vmatrix} \qquad (1.11)$$

See form (1.6), (1.7), (1.8), (1.9), (1.10), (1.11) following can be obtained:

$$A = 2\prod_{i=1}^{n}(x_i - x_0)^3 \begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^{3n} \\ & & & & \cdots & \\ 1 & x_{n-1} & x_{n-1}^2 & x_{n-1}^3 & \cdots & x_{n-1}^{3n} \\ 1 & x_n & x_n^2 & x_n^3 & \cdots & x_n^{3n} \\ 1 & x_0 & x_0^2 & x_0^3 & \cdots & x_0^{3n} \\ 0 & 1 & 2x_1 & 3x_1^2 & \cdots & 3nx_1^{3n-1} \\ 0 & 1 & 2x_2 & 3x_2^2 & \cdots & 3nx_2^{3n-1} \\ & & & & \cdots & \\ 0 & 1 & 2x_n & 3x_n^2 & \cdots & 3nx_n^{3n-1} \\ 0 & 0 & \frac{2!}{0!} & \frac{3!}{1!}x_0 & \cdots & \frac{3n!}{(3n-2)!}x_0^{3n-2} \\ 0 & 0 & \frac{2!}{0!} & \frac{3!}{1!}x_1 & \cdots & \frac{3n!}{(3n-2)!}x_1^{3n-2} \\ & & & & \cdots & \\ 0 & 0 & \frac{2!}{0!} & \frac{3!}{1!}x_n & \cdots & \frac{3n!}{(3n-2)!}x_n^{3n-1} \end{vmatrix}$$

(1.12)

Repeat to the method obtained form (1.12), from form (1.4).In determinants of form (1.11) ,the elements in rows n+1,and rows 2n+1, are deleted ,that result is

$$A = (-1)^{3n+1+1} 2\prod_{i=1}^{n}(x_i - x_0)^3 D \quad (1.13)$$

$$= (-1)^{2n} 2\prod_{i=1}^{n}(x_i - x_0)^3 (-1)^{n+1+1} D \quad (1.14)$$

Where D is

$$D=\begin{vmatrix}1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^{3n-1}\\ 1 & x_2 & x_2^2 & x_2^3 & \cdots & x_2^{3n-1}\\ \\ 1 & x_{n-1} & x_{n-1}^2 & x_{n=1}^3 & \cdots & x_{n-1}^{3n-1}\\ 1 & x_n & x_n^2 & x_n^3 & \cdots & x_n^{3n-1}\\ 0 & 1 & 2x_1 & 3x_1^2 & \cdots & (3n-1)x_1^{3n-2}\\ 0 & 1 & 2x_2 & 3x_2^2 & \cdots & (3n-1)x_2^{3n-2}\\ & & & & \cdots & \\ 0 & 1 & 2x_n & 3x_n^2 & \cdots & (3n-1)x_n^{3n-2}\\ 0 & 0 & \frac{2!}{0!} & \frac{3!}{1!}x_1 & \cdots & \frac{(3n-1)!}{(3n-3)!}x_1^{3n-3}\\ 0 & 0 & \frac{2!}{0!} & \frac{3!}{1!}x_2 & \cdots & \frac{(3n-2)!}{(3n-4)!}x_2^{3n-3}\\ & & & & \cdots & \\ \\ 0 & 0 & \frac{2!}{0!} & \frac{3!}{1!}x_n & \cdots & \frac{(3n-2)!}{(3n-4)!}x_n^{3n-3}\end{vmatrix} \qquad (1.15)$$

According the definition of algebra cofactor $(-1)^{n+1+1}D = D(3n+1)_{n+1,1}$

Therefore forms (1.13)and (1.5) can be rewritten to be

$$A = (-1)^{2n}2\prod_{i=1}^{n}(x_i - x_0)^9 D_{n+1,1.} \qquad (1.16)$$

And

$$D(3n+4)_{n+2,1} = (-1)^{3n+1}\prod(x_1 - x_0)^9 D(3n+1)_{n+1,1} \qquad (1.17)$$

Successively apply above method, the order of $D(3n+1)_{n+1.1}$ can be reduced, and then following lemma 1.2 can be obtained.

Lemma 1.2

$$D(3n+4)_{n+2.1} = (-1)^{\frac{1}{2}(3n^2+5n+6)}2^{n+1}\prod_{n\ge i>j\ge 0}(x_i - x_j)^9, \qquad (1.18)$$

where D(3n+4)$_{n+2.1}$ is the algebra cofactor dispelled the entries of n+2-th row、1st column in determinant D(3n+4) in form (1.2)

Proof: According to lemma 1.1,the algebra cofactor is $D(3n+1)_{n+1,1}$ following form :

$$D(3n+1)_{n+1,1}=(-1)^{n+1+1}\begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^{3n-1} \\ 1 & x_2 & x_2^2 & x_2^3 & \cdots & x_2^{3n-1} \\ & & & & & \\ 1 & x_{n-1} & x_{n-1}^2 & x_{n=1}^3 & \cdots & x_{n-1}^{3n-1} \\ 1 & x_n & x_n^2 & x_n^3 & \cdots & x_n^{3n-1} \\ 0 & 1 & 2x_1 & 3x_1^2 & \cdots & (3n-1)x_1^{3n-2} \\ 0 & 1 & 2x_2 & 3x_2^2 & \cdots & (3n-1)x_2^{3n-2} \\ & & & & \cdots & \\ 0 & 1 & 2x_n & 3x_n^2 & \cdots & (3n-1)x_n^{3n-2} \\ 0 & 0 & \frac{2!}{0!} & \frac{3!}{1!}x_1 & \cdots & \frac{(3n-1)!}{(3n-3)!}x_1^{3n-3} \\ 0 & 0 & \frac{2!}{0!} & \frac{3!}{1!}x_2 & \cdots & \frac{(3n-2)!}{(3n-4)!}x_2^{3n-3} \\ & & & & \cdots & \\ 0 & 0 & \frac{2!}{0!} & \frac{3!}{1!}x_n & \cdots & \frac{(3n-2)!}{(3n-4)!}x_n^{3n-3} \end{vmatrix} \quad (1.19)$$

In form (1.5),the order of determinants is $3n$, in which the elements contain x_0 have been dispelled ,by apply above method.Similarly, the elements contain x_i ($i=1,2,...,n-1.$)have been dispelled ,the determonant of order $3n$ can be reduced to be order 3. This results is :

$$D(3n+4)_{n+2,1}=(-1)^{n+2}(-1)^{\frac{1}{2}(3n^2-3n-4)}2^n\prod_{n\geq i>j\geq 0}^{n}(x_i-x_j)^9\begin{vmatrix} 1 & x_n & x_n^2 \\ 0 & 1 & 2x_n \\ 0 & 0 & \frac{2!}{0!} \end{vmatrix}$$

$$=(-1)^{\frac{1}{2}(3n^2+5n+6)}2^{n+1}\prod_{n\geq i>j\geq 0}^{n}(x_i-x_j)^9.$$

Lemma 1.3

$$D(3n+4)_{n+3+k,1}=(-1)^{3(n-1)}2(x-x_0)^3(x_k-x_0)^6\prod_{\substack{i=1\\ i\neq k}}^{n}(x_i-x_0)^9[D(3n+1)_{n+1+k,1}$$
$$-\frac{3!}{x_k-x_0}D(3n+1)_{2n+1+k,1}] \qquad (1.20)$$

Lemma 1.4

$$D(3n+4)_{1+k,1}=(-1)^{3n+1}2(x-x_0)^3(x_k-x_0)^6\prod_{\substack{i=1\\ i\neq k}}^{n}(x_i-x_0)^9[D(3n+1)_{k,1}$$
$$-\frac{3}{x_k-x_0}D(3n+1)_{n+1+k,1}+\frac{2\cdot 3!}{(x_k-x_0)^2}D(3n+1)_{2n+1+k,1}]. \qquad (1.21)$$

Lemma 1.5

$$D(3n+4)_{2n+4+k,1}=(-1)^{3n+5}2(x-x_0)^3(x_k-x_0)^6\prod_{\substack{i=1\\ i\neq k}}^{n}(x-x_0)^9 D(3n+1)_{2n+1+k,1}. \qquad \mathbf{(1.22)}$$

2.1 Main results

Following theorems are the main results of this paper:

Theorem 2.1

$$\frac{D(3n+4)_{2n+4+k,1}}{D(3n+4)_{n+2,1}}=-l_k''(n+1), \qquad (2.1)$$

where $D(3n+4)_{2n+4+k,1}$、 $D(3n+4)_{n+2,1}$ are the cofactors of $D(3n+4)$ in form (1.2) and $l_k''(n+1)$ is following form :

$$l_k''(n+1)=\frac{1}{2}(x-x_k)^2\prod_{\substack{i=0\\ i\neq k}}^{n}\left(\frac{x-x_i}{x_k-x_i}\right)^3 \qquad （2.2）$$

Proof . At first, form (2.1) can be verified as n=2, 3, and then, it is supposed to be correct as n:

$$\frac{D(3n+1)_{2n+1+k,1}}{D(3n+1)_{n+1,1}} = -l_k^{"}(n). \qquad (2.3)$$

where $D(3n+1)_{2n+1+k,1}$、 $D(3n+1)_{n+1,1}$ are the cofactors of D(3n+1) in form(1.3) and

$$l_k^{"}(n) = \frac{1}{2}(x-x_k)^2 \prod_{\substack{i=1 \\ i \neq k}}^{n} (\frac{x-x_i}{x_k-x_i})^3 . \qquad (2.4)$$

To prove it is correct as n+1, let $D(3n+4)_{2n+4+k,1}$ be divided by $D(3n+4)_{n+2,1}$ and notice form （1.4）(1.7), we have:

$$\frac{D(3n+4)_{2n+4+k,1}}{D(3n+4)_{n+2,1}} = \frac{(-1)^{3n+5} 2(x-x_0)^3 (x_k-x_0)^6 \prod_{\substack{n=1 \\ i \neq k}}^{n} (x_i-x_0)^9 D(3n+1)_{2n+1+k.1}}{(-1)^{3n+1} 2 \prod_{i=1}^{n} (x_i-x_0)^9 D(3n+1)_{n+1.1}}$$

$$= (\frac{x-x_0}{x_k-x_0})^3 \frac{D(3n+1)_{2n+1+k.1}}{D(3n+1)_{n+11}} . \qquad (2.5)$$

The right hand is replaced by form (2.3), and notice form (2.4), and (2.2), following form holds:

$$\frac{D(3n+4)_{2n+4+k.1}}{D(3n+4)_{n+2.1}} = -(\frac{x-x_0}{x_k-x_0})^3 l_k^{"}(n)$$

$$= -(\frac{x-x_0}{x_k-x_0})^3 [\frac{1}{2}(x-x_k)^2 \prod_{\substack{i=1 \\ i \neq k}}^{n} (\frac{x-x_i}{x_k-x_i})^3]$$

$$= -\frac{1}{2}(x-x_k)^2 \prod_{\substack{i=0 \\ i \neq k}}^{n} (\frac{x-x_i}{x_k-x_i})^3 = l_k^{"}(n+1).$$

That theorem is correct. |

See above the first, fourth equations we have:

$$(\frac{x-x_0}{x_k-x_0})^3 l_k''(n) = l_k''(n+1). \qquad (2.5')$$

Theorem 2.2

$$\frac{D(3n+4)_{n+3+k,1}}{D(3n+4)_{n+2,1}} = -l'_k(n+1) \qquad (2.6)$$

where D(3n+4)$_{n+3+k,1}$、D(3n+4)$_{n+2,1}$ are the cofactor of D(3n+4) in form (1.2), and $l_k^{'}(n+1)$ is following form ;

$$l'(n+1) = (x-x_k)[1+3(x-x_k)\sum_{\substack{i=0\\i\neq k}}^{n}\frac{1}{x_i-x_k}]\prod_{\substack{i=0\\i\neq k}}^{n}(\frac{x-x_i}{x_k-x_i})^3 \qquad (2.7)$$

Proof . The proof is as same as that of theorem2.1, however, there is some difference in ease of n+1. Now suppose it be correct as case of n:

$$\frac{D(3n+1)_{n+1+k.1}}{D(3n+1)_{n+1.1}} = -l'_k(n), \qquad (2.8)$$

where $D(3n+1)_{n+1+k.1}$、$D(3n+1)_{n+1,1}$ are the cofactors of D(3n+1) in form(2.3), and $l_k^{'}(n)$ is following form:

$$l'_k(n) = (x-x_k)[1+3(x-x_k)\sum_{\substack{i=1\\i\neq k}}^{n}\frac{1}{x_i-x_k}]\prod_{\substack{i=1\\i\neq k}}^{n}(\frac{x-x_i}{x_k-x_i})^3, \qquad (2.9)$$

and then form (1.4)、(1.5) are noticed, we have:

$$\frac{D(3n+4)_{n+3+k.1}}{D(3n+4)_{n+2.1}} = (\frac{x-x_0}{x_k-x_0})^3[\frac{D(3n+1)_{n+1+k1}}{D(3n+1)_{n+1.1}} - \frac{3!}{x_k-x_0}\frac{D(3n+1)_{2n+1+k.1}}{D(3n+1)_{n+1.1}}]$$

(2.10)

The right hand is replaced by form (2.7)、(2.3) ,and (2.9),(2.4),and then following form holds:

$$\frac{D(3N+4)_{n+3+k,1}}{D(3n+4)_{n+2,1}} = (\frac{x-x_0}{x_k-x_0})^3[-l_k'(n) + \frac{3!}{x_k-x_0}l_k''(n)] \qquad (2.11)$$

$$\frac{D(3N+4)_{n+3+k,1}}{D(3n+4)_{n+2,1}} = -(\frac{x-x_0}{x_k-x_0})^3(x-x_k)[1+3(x-x_k)\sum_{\substack{i=1\\i\neq k}}^{n}\frac{1}{x_i-x_k}] - \frac{3(x-x_k)}{x_k-x_0}\prod_{\substack{i=1\\i\neq k}}^{n}(\frac{x-x_i}{x_k-x_0})^3$$

$$= -(x-x_k)[1+3(x-x_k)\sum_{\substack{i=1\\ i\neq k}}^{n}\frac{1}{x_i-x_k}+\frac{3(x-x_k)}{x_0-x_k}]\prod_{\substack{i=0\\ i\neq k}}^{n}(\frac{x-x_i}{x_k-x_0})^3$$

$$= -(x-x_k)[1+3(x-x_k)\sum_{\substack{i=0\\ i\neq k}}^{n}\frac{1}{x_i-x_k}]\prod_{\substack{i=1\\ i\neq k}}^{n}(\frac{x-x_i}{x_k-x_0})^3$$

$$= -l_k'(n+1). \qquad (2.12)$$

So form (2.6) is correct.|

By the way, following form can be obtained, see form (2.11) and (2.12)

$$(\frac{x-x_0}{x_k-x_0})^3[l_k'(n)-\frac{3!}{x_k-x_0}l_k''(n)]=l_k'(n+1). \qquad (2.13)$$

Theorem 2.3

$$\frac{D(3n+4)_{1+k.1}}{D(3n+4)_{n+2.1}}=-l_k(n+1), \qquad (2.14)$$

where $D(3n+4)_{1+k.1}$、$D(3n+4)_{n+2.1}$ is the cofactor of D(3n+1) in form (1.3) and $l_k(n+1)$ is following form :

$$l_k(n+1)=[1+3(x-x_k)\sum_{\substack{i=0\\ i\neq k}}^{n}\frac{1}{x_i-x_k}+2g3(x-x_k)^2\sum_{\substack{i=0\\ i\neq k}}^{n}\frac{1}{(x_i-x_k)^2}$$

$$+3g3(x-x_k)^2\sum_{\substack{i=0\\ i\neq k}}^{n-1}\frac{1}{x_i-x_k}\sum_{j=i+1}^{n}\frac{1}{x_j-x_k}]\prod_{\substack{i=0\\ i\neq k}}^{n}(\frac{x-x_i}{x_k-x_i})^3 \qquad (2.15).$$

Proof . It is easy to verified in cases n=2、3，and supposed to be correct as case n, namely

$$\frac{D(3n+1)_{k,1}}{D(3n+1)_{n+1.1}}=-l_k(n), \qquad (2.16)$$

where $D(3n+1)_{k.1}$ and $D(3n+1)_{n+1.1}$ are the cofactors of D(3n+1) in form (1.3), and $l_k(n)$) is given by

$$l_k(n)=[1+3(x-x_k)\sum_{\substack{i=1\\ i\neq k}}^{n}\frac{1}{x_i-x_k}+2\cdot 3(x-x_k)^2\sum_{\substack{i=1\\ i\neq k}}^{n}\frac{1}{(x_i-x_k)^2}$$

$$+3\cdot 3(x-x_k)^2\sum_{\substack{i=1\\ i\neq k}}^{n-1}\frac{1}{x_i-x_k}\sum_{j=i+1}^{n}\frac{1}{x_i-x_k}]\prod_{\substack{i=1\\ i\neq k}}^{n}(\frac{x-x_i}{x_k-x_i})^3, \qquad (2.17)$$

Consider (1.4) and (1.6), we have:

$$\frac{D(3n+4)_{1+k.1}}{D(3n+4)_{n+2.1}}=(\frac{x-x_0}{x_k-x_0})^3\left[\frac{D(3n+1)_{k,1}}{D(3n+1)_{n+1,1}}-\frac{3}{x_k-x_0}\frac{D(3n+1)_{n+1+k,1}}{D(3n+1)_{n+1,1}}\right]$$

$$+\frac{2\cdot 3!}{(x_k-x_0)^2}\frac{D(3n+1)_{2n+1+k,1}}{D(3n+1)_{n+1,1}} \qquad (2.18)$$

In the right hand, the 1st, 2nd and 3rd items are replaced by form (2.16)、(2.8)、and (2.3) respectively, and notice (2.17),(2.9),(2.2), following form holds:

$$\frac{D(3n+4)_{1+k.1}}{D(3n+4)_{n+2.1}}=-(\frac{x-x_0}{x_k-x_0})^3[l_k(n)-\frac{3}{x_k-x_0}l_k^{'}(n)+\frac{2\cdot 3!}{(x_k-x_0)^2}l_k^{''}(n)]$$

$$=-(\frac{x-x_0}{x_k-x_0})^3\{[1+3(x-x_k)\sum_{\substack{i=1\\i\neq k}}^{n}\frac{1}{x_i-x_k}+2\cdot 3(x-x_k)^2\sum_{\substack{i=1\\i\neq j}}^{n}(\frac{1}{x_i-x_k})^2$$

$$+3\cdot 3(x-x_0)^2(\sum_{\substack{i=1\\i\neq k}}^{n}\frac{1}{x_i-x_k}\sum_{j=i+1}^{n}\frac{1}{x_j-x_k})]\prod_{\substack{i=1\\i\neq k}}^{n}(\frac{x-x_i}{x_k-x_i})^3$$

$$-\frac{3}{x_k-x_0}(x-x_0)[1+3(x-x_k)\sum_{\substack{i=1\\i\neq k}}^{n}\frac{1}{x_i-x_k}]\prod_{\substack{i=\\i\neq k}}^{n}(\frac{x-x_i}{x-x_k})^3$$

$$+[\frac{2\cdot 3!}{(x-x_0)^2}\cdot\frac{(x-x_0)^2}{2}]\prod_{\substack{i=1\\i\neq k}}^{n}(\frac{x-x_0}{x_k-x_i})^3\} \qquad (2.19)$$

Extract out the common factor $\prod_{\substack{i=1\\i\neq k}}^{n}(\frac{x-x_i}{x_k-x_i})^3$, which is on the outside of a square bracket and merge each item, which is on the inside of a square bracket by using following method :

$$3(x-x_k)\sum_{\substack{i=1\\i\neq k}}^{n}\frac{1}{x_i-x_k}-\frac{3}{x_k-x_0}(x-x_k)=3(x-x_k)\sum_{\substack{i=0\\i\neq k}}^{n}\frac{1}{x_i-x_k},$$

$$2\cdot 3(x-x_k)^2\sum_{\substack{i=1\\i\neq k}}^{n}(\frac{1}{x_i-x_k})^2+\frac{2\cdot 3!}{(x_k-x_0)^2}\cdot\frac{(x-x_0)^2}{2}=2\cdot 3(x-x_k)^2\sum_{\substack{i=0\\i\neq k}}^{n}(\frac{1}{x_i-x_k})^2$$

$$3g3(x-x_k)^2(\sum_{\substack{i=1\\ i\neq k}}^{n-1}\frac{1}{x_i-x_k}\sum_{j=i+1}^{n}\frac{1}{x_i-x_k})-\frac{3g3}{x_k-x_0}(x-x_k)^2\sum_{\substack{i=1\\ i\neq k}}^{n}\frac{1}{x_i-x_k}$$

$$=3g3(x-x_k)^2(\sum_{\substack{i=1\\ i\neq k}}^{n-1}\frac{1}{x_i-x_k}\sum_{j=i+1}^{n}\frac{1}{x_j-x_k}+\frac{1}{x_0-x_k}\sum_{\substack{i=1\\ i\neq k}}^{n}\frac{1}{x_i-x_k})$$

$$=3g3(x-x_k)^2(\sum_{\substack{i=0\\ i\neq k}}^{n-1}\frac{1}{x_i-x_k}\sum_{\substack{j=i+1\\ j\neq k}}^{n}\frac{1}{x_j-x_k}).$$

That results in:

$$\frac{D(3n+4)_{1+K.1}}{D(3n+4)_{n+2.1}}=-(\frac{x-x_0}{x_k-x_0})^3[1+3(x-x_k)\sum_{\substack{i=0\\ i\neq k}}^{n}\frac{1}{x_i-x_k}+2\cdot 3(x-x_k)^2\sum_{\substack{i=0\\ i\neq k}}^{n}(\frac{1}{x_i-x_k})^2$$

$$+3\cdot 3(x-x_k)^2(\sum_{\substack{i=0\\ i\neq k}}^{n}(\frac{1}{x_i-x_k})\sum_{\substack{j=i+1\\ j\neq k}}^{n}(\frac{1}{x_j-x_k})\prod_{\substack{i=1\\ i\neq k}}^{n}(\frac{x-x_i}{x_k-x_i})^3]$$

$$=-[1+3(x-x_k)\sum_{\substack{i=0\\ i\neq k}}^{n}(\frac{1}{x_i-x_k})+2\cdot 3(x-x_k)^2\sum_{\substack{i=0\\ i\neq k}}^{n}(\frac{1}{x_i-x_k})^2 \qquad (2.20)$$

$$+3\cdot 3(x-x_k)^2\sum_{\substack{i=0\\ i\neq k}}^{n-1}(\frac{1}{x_i-x_k})\sum_{\substack{j=i+1\\ j\neq k}}^{n}\frac{1}{x_j-x_k}]\prod_{\substack{i=0\\ i\neq k}}^{n}(\frac{x-x_i}{x_k-x_i})^3$$

$$=-l_k(n+1).$$

where $l_k(n+1)$ is from (2.15). It means form (2.14) is correct in case n+1.|

See the lst equation of form (2.19) and 3rd equations, of form (2.20) , following equation can be obtained:

$$(\frac{x-x_0}{x_k-x_0})^3[l_k(n)-\frac{3}{x_k-x_0}l_k^{'}(n)+\frac{2\cdot 3!}{(x_k-x_0)^2}l_k^{''}(n)]=l_k(n+1). \qquad (2.21)$$

Conclusion

The main results of this paper are the theorems 2.1, 2.2, 2.3. They are the computation of high order determinants and their elements consist of 0,1,2 order differentials whose elements of roads from 1 to n+1 in the determinants are the 0 order differentials of

$x_i(i=0,1,2...)$.and the elements of roads from n+2 to 2n+2 in the determinants are the 1 order differentials of $x_i(i=0,1,2...)$,and the elements of roads from 2n+3 to 3n+3 in the determinants are the 2 order differentials of $x_i(i=0,1,2...)$.Because of the high order ,to compute them are difficulty. But the have been reduced to the sum of several determinants which are lower-order. The several determinants of low order are easy to be computed .These results can be applied in interpolation method[4], which is important tool in modeling and simulation .For example the design the out line of air plane ,bigger ship and satellite need interpolation But the solution of the problem have not been seen in papers [9-16]. The theorems 2.1 2.2 2.3 are the basic of solution for the problem. Subject the limit of paper page , the more results will appear in our other papers.

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