AMSE JOURNALS –2014-Series: Advances A; Vol. 51; N°1 ; pp 80-99
Submitted April 2014; Revised Dec. 26, 2014; Accepted Jan. 20, 2015

The Construction of the Sensitivity Functionals in the Bolts's Problem for Multivariate Dynamic Systems described by Integro-Differential Equations with Delay Time

*A.I. Rouban

* Computer Science Dept, Institute of Space and Information Technologies,
Siberian Federal University, Street of academician Kirenskogo, number 26, Krasnoyarsk region,
Krasnoyarsk 660 074, Russia (ai-rouban@mail.ru)

Abstract

The variational method of calculation of sensitivity functional (connecting first variation of quality functional with variations of variable and constant parameters) for multivariate non-linear dynamic systems, described by Volterra's integro-differential equations of the second genus with delay time, by use of the generalized quality functional of a dynamic system (the Bolts problem) is developed. The presence of a discontinuity in an initial value of coordinates and association of initial, final instants and magnitude of delay from constant parameters is taken into account also. The base of calculation is the decision of corresponding integro-differential conjugate equations for Lagrange's multipliers in the opposite direction of time.

Key words: Variational method, sensitivity functional, integro-differential equation, conjugate equation, delay time.

1. Introduction

The problem of calculation of sensitivity functional (SF) (connecting first variation of quality functional with variations of variable and constant parameters) and sensitivity coefficients (SC) (components of vector gradient from quality functional according to constant parameters) of dynamic systems are main at the analysis and syntheses of control laws, identification, optimization [1–7]. The first-order sensitivity indexes are most used. Later on we shall examine only SF and SC of the first-order.

Consider a vector output $x(t)$ of dynamic object model under continuous time $t \in [t_0, t^1]$, implicitly depending on vectors parameters $\tilde{\alpha}(t), \overline{\alpha}$, and functional I constructed on the basis of $x(t)$ under $t \in [t_0, t^1]$. The first variation δI and variations $\delta\tilde{\alpha}(t)$ are connected with each other with the help of a single-line functional – SF with respect to variable parameters $\tilde{\alpha}(t)$: $\delta_{\tilde{\alpha}(t)} I = \int_{t_0}^{t^1} V(t)\delta\tilde{\alpha}(t)dt$. SC with respect to constant parameters $\overline{\alpha}$ are called a gradient from I on $\overline{\alpha}$: $(dI/d\overline{\alpha})^T \equiv \nabla_{\overline{\alpha}} I$. SC are a coefficients of single-line relationship between the first variation of functional δI and the variations of constant parameters $\overline{\alpha}$: $\delta_{\overline{\alpha}} I = (dI/d\overline{\alpha})\delta\overline{\alpha} \equiv \sum_{j=1}^{m} (\partial I/\partial\overline{\alpha}_j)\delta\overline{\alpha}_j$.

For simplest classes of dynamic systems it is shown, that at the SC calculation it is possible to pass from a solution of the bulky sensitivity equations to a solution of the conjugate equations – conjugated with respect to object dynamic equations. Method of receipt of conjugate equations (it was offered in 1962) is cumbersome, because it is based on the analysis of sensitivity equations, and it does not get its developments.

Variational method [4], ascending to Lagrange's, Hamilton's, Euler's memoirs, makes possible to simplify the process of determination of conjugate equations and formulas of account of SC and SF. On the basis of this method it is an extension of quality functional by means of including into it object dynamic equations by means of Lagrange's multipliers and obtaining the first variation of extended functional in phase coordinates of object and on interesting parameters. The dynamic equations for Lagrange's multipliers are obtained from a condition of equality to zero of appropriate components (relative to phase coordinates) of the first variation of an extended functional. Given simplification of first variation of extended functional brings at presence in the right part only parameter variations, i.e. it is produced the SF. If all parameters are constant then the parameters variations are carried out from corresponding integrals and at the final result in obtained functional variation the coefficients before parameters variations are the required SC. Given method was used in [8] for dynamic systems described by ordinary continuous Volterra's integral and integro-differential equations of the second genus.

In this paper the variational method of account of SF develops to more general continuous many-dimensional non-linear dynamic systems circumscribed by the vectorial non-linear continuous Volterra's integro-differential equations of the second genus with delay time, with variable $\tilde{\alpha}(t)$ and constant $\overline{\alpha}$ parameters and with reviewing of generalised quality functional (the Bolts problem) and registration of dependencies: 1) disturbing actions of a object model from initial instant; 2) of initial t_0 and final t^1 instants and of dead time from constant parameters $\overline{\alpha}$.

2. Statement of the problem

Consider that the dynamic object is described by system of non-linear continuous integro-differential equations (I-DE) with integral components of Volterra's type of the second genus with delay time τ [7, p. 75] and variable $\tilde{\alpha}(t)$ and constant $\overline{\alpha}$ parameters

$$\dot{x}(t) = f(x(t), x(t-\tau), y(t), y(t-\tau), \tilde{\alpha}(t), \overline{\alpha}, t), \ t_0 < t \le t^1, \ 0 < \tau, \tag{1}$$

$$y(t) = r(\tilde{\alpha}(t), \overline{\alpha}, t_0, t) + \int_{t_0}^{t} K(t, x(s), x(s-\tau), y(s), y(s-\tau), \tilde{\alpha}(s), \overline{\alpha}, s)\, ds, \ t_0 \le t \le t^1,$$

$$t_0 = t_0(\overline{\alpha}),\ t^1 = t^1(\overline{\alpha}),\ \tau = \tau(\overline{\alpha}),$$
$$x(t) = \psi_x(\tilde{\alpha}(t), \overline{\alpha}, t),\ t \in [t_0 - \tau, t_0),\ x(t_0) = x_0(\overline{\alpha}, t_0),$$
$$y(t) = \psi_y(\tilde{\alpha}(t), \overline{\alpha}, t),\ t \in [t_0 - \tau, t_0).$$

Here: the magnitudes of initial t_0 and final t^1 instants and also dead time τ and initial values $x(t_0)$ are known functions from constant parameters $\overline{\alpha}$: $t_0 = t_0(\overline{\alpha}),\ t^1 = t^1(\overline{\alpha}),\ \tau = \tau(\overline{\alpha})$, $x(t_0) = x_0(\overline{\alpha}, t_0)$; x, y – a vector-columns of phase coordinates; $\tilde{\alpha}(t), \overline{\alpha}$ – vector-columns of interesting variable and constant parameters; $f(\cdot)$, $\psi_x(\cdot)$, $r(\cdot)$, $K(\cdot)$, $\psi_y(\cdot)$, $t_0(\overline{\alpha})$, $t^1(\overline{\alpha})$, $\tau(\overline{\alpha})$, $x_0(\cdot)$ – known continuously differentiated limited vector-functions. The phase coordinates x, y in an index point t_0 makes a discontinuity if:

$$x^+(t_0) \equiv x(t_0+0) \equiv x_0(\alpha, t_0) \ne x(t_0-0) \equiv x^-(t_0) \equiv \psi_x(\tilde{\alpha}(t_0), \overline{\alpha}, t_0),$$
$$y^+(t_0) \equiv y(t_0+0) \equiv r(\tilde{\alpha}(t_0), \overline{\alpha}, t_0, t_0) \ne y(t_0-0) \equiv y^-(t_0) \equiv \psi_y(\tilde{\alpha}(t_0), \overline{\alpha}, t_0).$$

But at the expense of an integration in a model (1) phase coordinates become continuous in instants $t_0 + n\tau,\ n = 1, 2, \cdots$. Here is designated: $x^+(t_0) \equiv x(t_0+0)$ – value of a phase coordinate to the right of a point t_0, and accordingly $x^-(t_0) \equiv x(t_0-0)$ – to the left of a point t_0.

The model of a measuring device is given as

$$\eta(t) = \eta(x(t), y(t), \tilde{\alpha}(t), \bar{\alpha}, t),\ t \in [t_0, t^1], \tag{2}$$

where $\eta(\cdot)$ is also continuous, differentiated, limited (together with the first derivatives) vector-function. The required parameters $\tilde{\alpha}(t), \bar{\alpha}$ are inserted also in (2). Dimensionality of vectors x, y and η in general event can be different.

On the basis of output coordinates of a measuring device η the quality functional of a dynamic system is constructed:

$$I(\alpha) = \int_{t_0}^{t^1} f_0(\eta(t), \tilde{\alpha}(t), \bar{\alpha}, t)\, dt + I_1(\eta(t^1), \bar{\alpha}, t^1). \tag{3}$$

Conditions for functions $f_0(\cdot)$, $I_1(\cdot)$ are the same as for $f(\cdot)$, $r(\cdot)$, $K(\cdot)$, $t_0(\cdot)$, $t^1(\cdot)$, $\tau(\cdot)$, $\psi_x(\cdot)$, $x_0(\cdot)$, $\psi_y(\cdot)$, etc.

With use of a functional (3) in the optimization problem (in the theory of optimal control) known as the Bolts's problem. From it as the individual variants follows: Lagrange's problem (when there is only integrated component) and Mayer's problem (when there is only second component – function from phase coordinates at a finishing point).

With the purpose of simplification of appropriate deductions with preservation of a generality in all transformations (1) – (3) there are a two vectors of parameters $\tilde{\alpha}(t), \bar{\alpha}$. If in the equations (1) – (3) the parameters are different then it is possible formally to unite them in one vector $\tilde{\alpha}(t)$ are $\bar{\alpha}$, to use obtained outcomes and then, taking into account a structure of a vectors $\tilde{\alpha}(t), \bar{\alpha}$, to make appropriate simplifications.

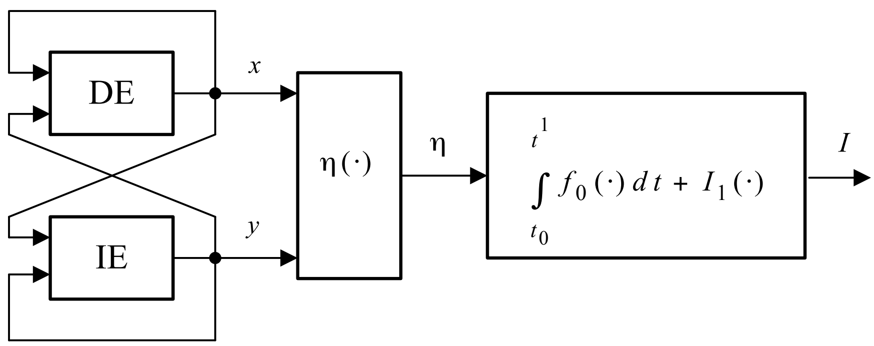

Fig. 1. The scheme of interaction between variables in (1) – (3)

The scheme of interaction between variables of object model, measuring device and quality functional is shown in fig. 1.

By obtaining of results the obvious designations:

$f(t) \equiv f(x(t), x(t-\tau), y(t), y(t-\tau), \tilde{\alpha}(t), \bar{\alpha}, t)$, $r(t) \equiv r(\tilde{\alpha}(t), \bar{\alpha}, t_0, t)$,

$K(t,s) \equiv K(t,x(s),x(s-\tau),y(s),y(s-\tau),\tilde{\alpha}(t),\bar{\alpha},s)$, $\eta(t) \equiv \eta(x(t),y(t),\tilde{\alpha}(t),\bar{\alpha},t)$,

$f_0(t) \equiv f_0(\eta(t),\tilde{\alpha}(t),\bar{\alpha},t),\ I_1(t^1) \equiv I_1(\eta(t^1),\bar{\alpha},t^1)$

are used.

Let's pass from the integro-differential equations (I-DE) to integral equations (IE), we shall generalize results of paper [9] on more general models (15), (16) with variables and constant parameters $\tilde{\alpha}(t),\bar{\alpha}$ and then we shall return to initial variables (see 6. Appendix).

The convenience of a integro-differential model consists in its structural universality. At simplification of a model it is enough in final results to convert in a zero appropriate addends. This reception we shall apply in a final part of a paper.

3. Basic result

Conjugate equations have the form

$$-\dot{\lambda}_x^T(t) = \lambda_x^T(t)\frac{\partial f(t)}{\partial x(t)} + \Phi_y(t^1)\frac{\partial K(t^1,t)}{\partial x(t)} + \frac{\partial f_0(t)}{\partial \eta(t)}\frac{\partial \eta(t)}{\partial x(t)} +$$

$$+\int_t^{t^1}\gamma_y^T(s)\frac{\partial K(s,t)}{\partial x(t)}ds + 1(t^1-\tau-t)[\lambda_x^T(t+\tau)\frac{\partial f(t+\tau)}{\partial x(t)} +$$

$$+\Phi_y(t^1)\frac{\partial K(t^1,t+\tau)}{\partial x(t)} + \int_{t+\tau}^{t^1}\gamma_y^T(s)\frac{\partial K(s,t+\tau)}{\partial x}ds\ ],\ t\in[t_0,t^1),$$

$$\lambda_x^T(t^1) = \Phi_x(t^1),$$

$$\gamma_y^T(t) = \lambda_x^T(t)\frac{\partial f(t)}{\partial y(t)} + \Phi_y(t^1)\frac{\partial K(t^1,t)}{\partial y(t)} + \frac{\partial f_0(t)}{\partial \eta(t)}\frac{\partial \eta(t)}{\partial y(t)} + \tag{4}$$

$$+\int_t^{t^1}\gamma_y^T(s)\frac{\partial K(s,t)}{\partial y(t)}ds + 1(t^1-\tau-t)[\lambda_x^T(t+\tau)\frac{\partial f(t+\tau)}{\partial y(t)} +$$

$$+\Phi_y(t^1)\frac{\partial K(t^1,t+\tau)}{\partial y(t)} + \int_{t+\tau}^{t^1}\gamma_y^T(s)\frac{\partial K(s,t+\tau)}{\partial y}ds\ ],\ t_0 \le t \le t^1;$$

$$\bar{\gamma}_x^T(t) = 1(t^1-\tau-t)[\lambda_x^T(t+\tau)\frac{\partial f(t+\tau)}{\partial x(t)} + \Phi_y(t^1)\frac{\partial K(t^1,t+\tau)}{\partial x(t)} + \int_{t+\tau}^{t^1}\gamma_y^T(s)\frac{\partial K(s,t+\tau)}{\partial x(t)}ds\ ],$$

$$\bar{\gamma}_y^T(t) = 1(t^1-\tau-t)[\lambda_x^T(t+\tau)\frac{\partial f(t+\tau)}{\partial y(t)} + \Phi_y(t^1)\frac{\partial K(t^1,t+\tau)}{\partial y(t)} +$$

$$+\int_{t+\tau}^{t^1} \gamma_y^T(s)\frac{\partial K(s,t+\tau)}{\partial y(t)}ds\ ],\ t_0-\tau \le t \le t_0.$$

Here $\Phi_x(t^1) \equiv \frac{\partial I_1(t^1)}{\partial \eta(t^1)}\frac{\partial \eta(t^1)}{\partial x(t^1)}$, $\Phi_y(t^1) \equiv \frac{\partial I_1(t^1)}{\partial \eta(t^1)}\frac{\partial \eta(t^1)}{\partial y(t^1)}$, $1(z)$ – is a single function: $1(z) = 1$ if $0 < z$, $1(z) = 0$ if $z \le 0$.

SF are calculated under the formula:

$$\delta I = \delta_{\tilde\alpha(t)}I + \delta_{\tilde\alpha(t^1)}I + \delta_{\bar\alpha}I; \tag{5}$$

$$\delta_{\tilde\alpha(t)}I = \int_{t_0}^{t^1}\left[\frac{\partial f_0(t)}{\partial \eta(t)}\frac{\partial \eta(t)}{\partial \tilde\alpha(t)} + \frac{\partial f_0(t)}{\partial \tilde\alpha(t)} + \gamma_y^T(t)\frac{\partial r(t)}{\partial \tilde\alpha(t)} + \lambda_x^T(t)\frac{\partial f(t)}{\partial \tilde\alpha(t)} + \right.$$

$$\left. + \Phi_y(t^1)\frac{\partial K(t^1,t)}{\partial \tilde\alpha(t)} + \int_t^{t^1}\gamma_y^T(s)\frac{\partial K(s,t)}{\partial \tilde\alpha(t)}ds\right]\delta\tilde\alpha(t)\,dt +$$

$$+\int_{t_0-\tau}^{t_0}[\bar\gamma_x^T(t)\frac{\partial \psi_x(t)}{\partial \tilde\alpha(t)} + \bar\gamma_y^T(t)\frac{\partial \psi_y(t)}{\partial \tilde\alpha(t)}]\delta\tilde\alpha(t)\,dt;$$

$$\delta_{\tilde\alpha(t^1)}I = [\frac{\partial I_1(t^1)}{\partial \eta(t^1)}\frac{\partial \eta(t^1)}{\partial \tilde\alpha(t^1)} + \Phi_y(t^1)\frac{\partial r(t^1)}{\partial \tilde\alpha(t^1)}]\delta\tilde\alpha(t^1);$$

$$\delta_{\bar\alpha}I = \left\{\frac{\partial I_1(t^1)}{\partial \eta(t^1)}\frac{\partial \eta(t^1)}{\partial \bar\alpha} + \frac{\partial I_1(t^1)}{\partial \bar\alpha} + \lambda_x^T(t_0)\frac{\partial x_0(\bar\alpha,t_0)}{\partial \bar\alpha} + \int_{t_0}^{t^1}\lambda_x^T(t)\frac{\partial f(t)}{\partial \bar\alpha}dt + \right.$$

$$+\Phi_y(t^1)\left[\frac{\partial r(t^1)}{\partial \bar\alpha} + \int_{t_0}^{t^1}\frac{\partial K(t^1,s)}{\partial \bar\alpha}ds\right] +$$

$$+\int_{t_0}^{t^1}[\frac{\partial f_0(t)}{\partial \eta(t)}\frac{\partial \eta(t)}{\partial \bar\alpha} + \frac{\partial f_0(t)}{\partial \bar\alpha} + \gamma_y^T(t)\frac{\partial r(t)}{\partial \bar\alpha} + \int_t^{t^1}\gamma_y^T(s)\frac{\partial K(s,t)}{\partial \bar\alpha}ds]dt +$$

$$+\int_{t_0-\tau}^{t_0}[\bar\gamma_x^T(t)\frac{\partial \psi_x(t)}{\partial \bar\alpha} + \bar\gamma_y^T(t)\frac{\partial \psi_y(t)}{\partial \bar\alpha}]dt +$$

$$+\left[\lambda_x^T(t_0)\left(\frac{\partial x_0(\bar\alpha,t_0)}{\partial t_0} - f(t_0)\right) + \right.$$

$$+1(t^1-t_0-\tau)\lambda_x^T(t_0+\tau)(f(t_0+\tau-0)-f(t_0+\tau+0))+$$

$$+\Phi_y(t^1)[\frac{\partial r(t^1)}{\partial t_0}-K(t^1,t_0)+1(t^1-t_0-\tau)(K(t^1,t_0+\tau-0)-K(t^1,t_0+\tau+0))]-$$

$$-f_0(t_0)+\int_{t_0}^{t^1}\gamma_y^T(t)(\frac{\partial r(t)}{\partial t_0}-K(t,t_0))dt+$$

$$\left.+1(t^1-t_0-\tau)\int_{t_0+\tau}^{t^1}\gamma_y^T(t)[K(t,t_0+\tau-0)-K(t,t_0+\tau+0)]dt\right]\frac{dt_0}{d\alpha}+$$

$$+\left[\Phi_x(t^1)f(t^1)+\Phi_y(t^1)[\frac{\partial r(t^1)}{\partial t^1}+K(t^1,t^1)+\int_{t_0}^{t^1}\frac{\partial K(t^1,s)}{\partial t^1}ds]+\right.$$

$$\left.+\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial t^1}+\frac{\partial I_1(t^1)}{\partial t^1}+f_0(t^1)\right]\frac{dt^1}{d\alpha}+$$

$$+\Big[1(t^1-t_0-\tau)\lambda_x^T(t_0+\tau)(f(t_0+\tau-0)-f(t_0+\tau+0))-$$

$$-\int_{t_0}^{t^1}\lambda_x^T(t)\left(\frac{\partial f(t)}{\partial x(t-\tau)}\frac{dx(t-\tau)}{d(t-\tau)}+\frac{\partial f(t)}{\partial y(t-\tau)}\frac{dy(t-\tau)}{d(t-\tau)}\right)dt+$$

$$+\Phi_y(t^1)[\,1(t^1-t_0-\tau)(K(t^1,t_0+\tau-0)-K(t^1,t_0+\tau+0))-$$

$$-\int_{t_0}^{t^1}\left(\frac{\partial K(t^1,s)}{\partial x(s-\tau)}\frac{dx(s-\tau)}{d(s-\tau)}+\frac{\partial K(t^1,s)}{\partial y(s-\tau)}\frac{dy(s-\tau)}{d(s-\tau)}\right)ds\,]+$$

$$+1(t^1-t_0-\tau)\int_{t_0+\tau}^{t^1}\gamma_y^T(t)(K(t,t_0+\tau-0)-K(t,t_0+\tau+0))dt-$$

$$\left.-\int_{t_0}^{t^1}\gamma_y^T(t)\int_{t_0}^{t}\left(\frac{\partial K(t,s)}{\partial x(s-\tau)}\frac{dx(s-\tau)}{d(s-\tau)}+\frac{\partial K(t,s)}{\partial y(s-\tau)}\frac{dy(s-\tau)}{d(s-\tau)}\right)ds\,dt\right]\frac{d\tau}{d\alpha}.$$

Derivation of these equations is represented in section 6.

The obtained form of representation of the conjugate equations and SF allows easily to write out outcomes with reviewing separately of differential and integrated models or their various combinations.

4. SF with use of a differential model for a object

The equations for an object model and measuring device look like:

$$\dot{x}(t) = f(x(t), x(t-\tau), \tilde{\alpha}(t), \overline{\alpha}, t), \;\; t_0 < t \le t^1, \;\; 0 < \tau,$$

$$t_0 = t_0(\overline{\alpha}), \;\; t^1 = t^1(\overline{\alpha}), \; \tau = \tau(\overline{\alpha}), \tag{6}$$

$$x(t) = \psi_x(\tilde{\alpha}(t), \overline{\alpha}, t), \; t \in [t_0 - \tau, t_0), \;\; x(t_0) = x_0(\overline{\alpha}, t_0);$$

$$\eta(t) = \eta(x(t), \tilde{\alpha}(t), \overline{\alpha}, t), \;\; t \in [t_0, t^1].$$

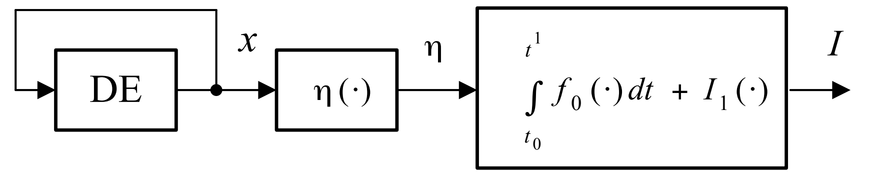

Fig. 2. The scheme of interaction between variables in (6), (3)

The form of a quality functional (3) does not vary. The scheme of interaction between all variables of a system is represented in fig. 2.

In total results (4), (5) it is necessary to take into account an absence in functions f, η dependence from $y(t)$ and also vanishing of functions r, K, ψ_y and variables $\gamma_y(t)$, $\overline{\gamma}_y(t)$:

$$-\dot{\lambda}_x^T(t) = \lambda_x^T(t)\frac{\partial f(t)}{\partial x(t)} + \frac{\partial f_0(t)}{\partial \eta(t)}\frac{\partial \eta(t)}{\partial x(t)} + 1(t^1 - \tau - t)\lambda_x^T(t+\tau)\frac{\partial f(t+\tau)}{\partial x(t)},$$

$$\lambda_x^T(t^1) = \Phi_x(t^1), \;\; t_0 \le t \le t, \tag{7}$$

$$\overline{\gamma}_x^T(t) = 1(t^1 - \tau - t)\lambda_x^T(t+\tau)\frac{\partial f(t+\tau)}{\partial x(t)}, \;\; t_0 - \tau \le t \le t_0.$$

SF has a form:

$$\delta I = \delta_{\tilde{\alpha}(t)} I + \delta_{\tilde{\alpha}(t^1)} I + \delta_{\overline{\alpha}} I; \tag{8}$$

$$\delta_{\tilde{\alpha}(t)} I = \int_{t_0}^{t^1}\left[\frac{\partial f_0(t)}{\partial \eta(t)}\frac{\partial \eta(t)}{\partial \tilde{\alpha}(t)} + \frac{\partial f_0(t)}{\partial \tilde{\alpha}(t)} + \lambda_x^T(t)\frac{\partial f(t)}{\partial \tilde{\alpha}(t)}\right]\delta\tilde{\alpha}(t)\,dt + + \int_{t_0-\tau}^{t_0} \overline{\gamma}_x^T(t)\frac{\partial \psi_x(t)}{\partial \tilde{\alpha}(t)}\delta\tilde{\alpha}(t)\,dt;$$

$$\delta_{\tilde{\alpha}(t^1)} I = \frac{\partial I_1(t^1)}{\partial \eta(t^1)}\frac{\partial \eta(t^1)}{\partial \tilde{\alpha}(t^1)}\delta\overline{\alpha}(t^1);$$

$$\delta_{\overline{\alpha}} I = \frac{\partial I_1(t^1)}{\partial \eta(t^1)}\frac{\partial \eta(t^1)}{\partial \overline{\alpha}} + \frac{\partial I_1(t^1)}{\partial \overline{\alpha}} + \lambda_x^T(t_0)\frac{\partial x_0(\overline{\alpha}, t_0)}{\partial \overline{\alpha}} + \int_{t_0}^{t^1}\lambda_x^T(t)\frac{\partial f(t)}{\partial \overline{\alpha}}dt +$$

$$+\int_{t_0}^{t^1}[\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial\overline{\alpha}}+\frac{\partial f_0(t)}{\partial\overline{\alpha}}]dt+\int_{t_0-\tau}^{t_0}\overline{\gamma}_x^T(t)\frac{\partial\psi_x(t)}{\partial\overline{\alpha}}dt+\left[\lambda_x^T(t_0)\left(\frac{\partial x_0(\overline{\alpha},t_0)}{\partial t_0}-f(t_0)\right)+\right.$$

$$\left.+1(t^1-t_0-\tau)\lambda_x^T(t_0+\tau)(f(t_0+\tau-0)-f(t_0+\tau+0))-f_0(t_0)\right]\frac{dt_0}{d\overline{\alpha}}+$$

$$+\left[\Phi_x(t^1)f(t^1)+\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial t^1}+\frac{\partial I_1(t^1)}{\partial t^1}+f_0(t^1)\right]\frac{dt^1}{d\overline{\alpha}}+$$

$$+\left[1(t^1-t_0-\tau)\lambda_x^T(t_0+\tau)(f(t_0+\tau-0)-f(t_0+\tau+0))-\right.$$

$$\left.-\int_{t_0}^{t^1}\lambda_x^T(t)\frac{\partial f(t)}{\partial x(t-\tau)}\frac{dx(t-\tau)}{d(t-\tau)}dt\right]\frac{d\tau}{d\overline{\alpha}}\Bigg\}\delta\overline{\alpha}.$$

We received more the general result in relation to results of publications [8], [9], [11], [12].

5. SF with use of variants of an integro-differential model

Variant 5.1. In initial statement of the task (1), (2) the equations of a model of object and measuring device vary:

$$\dot{x}(t)=f(x(t),x(t-\tau),\tilde{\alpha}(t),\overline{\alpha},t),\ t_0\le t\le t^1,\ 0<\tau,\quad x(t_0)=x_0(\overline{\alpha},t_0),$$

$$y(t)=r(\tilde{\alpha}(t),\overline{\alpha},t_0,t)+\int_{t_0}^{t}K(t,x(s),x(s-\tau),y(s),y(s-\tau),\tilde{\alpha}(s),\overline{\alpha},s)\,ds,\ t_0\le t\le t^1; \qquad (9)$$

$$t_0=t_0(\overline{\alpha}),\ t^1=t^1(\overline{\alpha}),\ \tau=\tau(\overline{\alpha}),$$

$$x(t)=\psi_x(\tilde{\alpha}(t),\overline{\alpha},t),\ t\in[t_0-\tau,t_0),\ x(t_0)=x_0(\overline{\alpha},t_0),$$

$$y(t)=\psi_y(\tilde{\alpha}(t),\overline{\alpha},t),\ t\in[t_0-\tau,t_0);$$

$$\eta(t)=\eta(y(t),\tilde{\alpha}(t),\overline{\alpha},t),\ t\in[t_0,t^1].$$

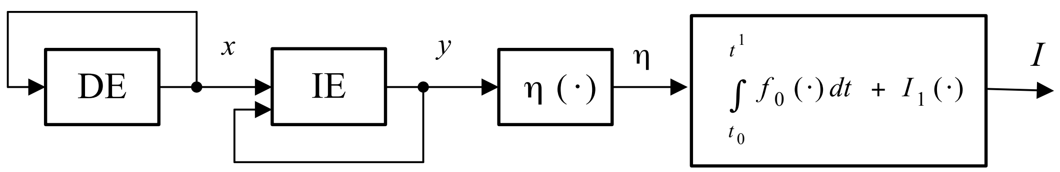

Fig. 3. The scheme of interaction between variables in (8), (3)

The scheme of interaction between variables of a object model, measuring device and quality functional is shown in fig. 3.

In total results (4), (5) it is necessary to take into account that $\partial\eta/\partial x = 0$, $\Phi_x(t^1) = 0$, $\partial f(t)/\partial y(t) = 0$, $\partial f(t+\tau)/\partial y(t) = 0$.

Variant 5.2. The equations of a object model and measuring device look like:

$$\dot{x}(t) = f(y(t), y(t-\tau), \tilde{\alpha}(t), \overline{\alpha}, t),\ t_0 \le t \le t^1,\ 0 < \tau,$$

$$y(t) = r(\tilde{\alpha}(t), \overline{\alpha}, t_0, t) + \int_{t_0}^{t} K(t, x(s), x(s-\tau), y(s), y(s-\tau), \tilde{\alpha}(s), \overline{\alpha}, s)\, ds,\ t_0 \le t \le t^1, \tag{10}$$

$$t_0 = t_0(\overline{\alpha}),\ t^1 = t^1(\overline{\alpha}),\ \tau = \tau(\overline{\alpha}),$$

$$x(t) = \psi_x(\tilde{\alpha}(t), \overline{\alpha}, t),\ t \in [t_0 - \tau, t_0),\ x(t_0) = x_0(\overline{\alpha}, t_0),$$

$$y(t) = \psi_y(\tilde{\alpha}(t), \overline{\alpha}, t),\ t \in [t_0 - \tau, t_0);$$

$$\eta(t) = \eta(y(t), \tilde{\alpha}(t), \overline{\alpha}, t),\ t \in [t_0, t^1].$$

The scheme of interaction between variables of a object model, measuring device and quality functional is represented in fig. 4.

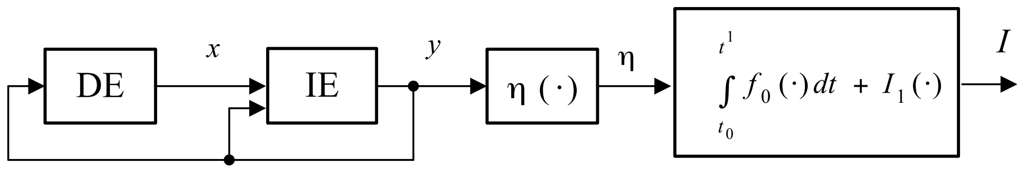

Fig. 4. The scheme of interaction between variables in (9), (3)

In total results (4), (5) it is necessary to take into account that $\partial\eta/\partial x = 0$, $\Phi_x(t^1) = 0$, $\partial f(t)/\partial x(t) = 0$, $\partial f(t+\tau)/\partial x(t) = 0$.

Variant 5.3. The equations of a object model and measuring device look like:

$$\dot{x}(t) = f(x(t), x(t-\tau), y(t), y(t-\tau), \tilde{\alpha}(t), \overline{\alpha}, t),\ t_0 \le t \le t^1,\ 0 < \tau,$$

$$y(t) = r(\tilde{\alpha}(t), \overline{\alpha}, t_0, t) + \int_{t_0}^{t} K(t, s, y(s), y(s-\tau), \tilde{\alpha}(s), \overline{\alpha})\, ds,\ t_0 \le t \le t^1,$$

$$t_0 = t_0(\overline{\alpha}),\ t^1 = t^1(\overline{\alpha}),\ \tau = \tau(\overline{\alpha}), \tag{11}$$

$$x(t) = \psi_x(\tilde{\alpha}(t), \overline{\alpha}, t),\ t \in [t_0 - \tau, t_0),\ x(t_0) = x_0(\overline{\alpha}, t_0),$$

$$y(t) = \psi_y(\tilde{\alpha}(t), \overline{\alpha}, t),\ t \in [t_0 - \tau, t_0);$$

$$\eta(t) = \eta(x(t), \tilde{\alpha}(t), \overline{\alpha}, t),\ t \in [t_0, t^1].$$

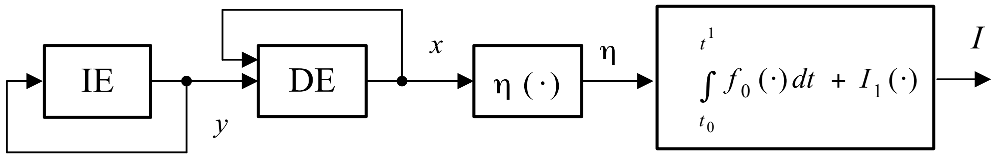

Fig. 5. The scheme of interaction between variables in (10), (3)

x is a basic variable. It satisfies to the differential equation. The variable y represents itself as input for a basic differential model and y satisfies to an independent integral equation. The scheme of interaction between variables is shown in fig. 5.

In total results (4), (5) it is necessary to take into account that $\partial\eta/\partial y = 0$, $\Phi_y(t^1) = 0$, $\partial K(s,t)/\partial x(t) = 0$, $\partial K(s,t+\tau)/\partial x(t) = 0$.

Variant 5.4. The equations of a object model have the form:

$$\dot{x}(t) = f(x(t), x(t-\tau), y(t), y(t-\tau), \tilde{\alpha}(t), \overline{\alpha}, t),\ t_0 < t \le t^1,\ \ 0 < \tau,$$

$$y(t) = r(\tilde{\alpha}(t), \overline{\alpha}, t_0, t) + \int_{t_0}^{t} K(t, x(s), x(s-\tau), \tilde{\alpha}(s), \overline{\alpha}, s)\, ds,\ t_0 \le t \le t^1,$$

$$\tau = \tau(\overline{\alpha}),\ t_0 = t_0(\overline{\alpha}),\ t^1 = t^1(\overline{\alpha}), \tag{12}$$

$$x(t) = \psi_x(\tilde{\alpha}(t), \overline{\alpha}, t),\ t \in [t_0 - \tau, t_0),\ x(t_0) = x_0(\overline{\alpha}, t_0),$$

$$y(t) = \psi_y(\tilde{\alpha}(t), \overline{\alpha}, t),\ t \in [t_0 - \tau, t_0).$$

The variable y is auxiliary. It reflects rather simple integrated connection from a basic variable x. The exit of a measuring device also depends only on a basic variable:

$$\eta(t) = \eta(x(t), \tilde{\alpha}(t), \overline{\alpha}, t),\ t \in [t_0, t^1].$$

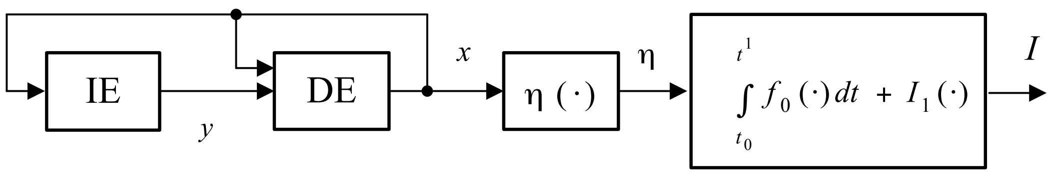

Fig. 6. The scheme of interaction between variables in (11), (12), (3)

The scheme of interaction between variables is represented in fig. 6.

In total results for conjugate equations (4) and SC (5) it is necessary to take into account that

$\partial\eta/\partial y = 0$, $\Phi_y(t^1) = 0$, $\partial K(s,t)/\partial y(t) = 0$, $\partial K(s,t+\tau)/\partial y(t) = 0$.

This article continues research in [8, 9, 11, 12].

6. Appendix

6.1. Passage to IE

In I-DE (1) the differential equation we write in the integral form

$$x(t)=x_0(\overline{\alpha},t_0)+\int_{t_0}^{t} f(x(s),x(s-\tau),y(s),y(s-\tau),\tilde{\alpha}(s),\overline{\alpha},s)ds,\ t_0\le t\le t^1. \tag{13}$$

We use notations

$$\tilde{y}(t)=\begin{pmatrix}x(t)\\ y(t)\end{pmatrix},\quad \tilde{r}(\tilde{\alpha}(t),\overline{\alpha},t_0,t)=\begin{pmatrix}x_0(\overline{\alpha},t_0)\\ r(\tilde{\alpha}(t),\overline{\alpha},t_0,t)\end{pmatrix}\equiv\begin{pmatrix}x_0(\overline{\alpha},t_0)\\ r(t)\end{pmatrix}\equiv\tilde{r}(t),$$

$$\tilde{K}(t,\tilde{y}(s),\tilde{y}(s-\tau),\tilde{\alpha}(s),\overline{\alpha},s)=\begin{pmatrix}f(x(s),x(s-\tau),y(s),y(s-\tau),\tilde{\alpha}(s),\overline{\alpha},s)\\ K(t,x(s),x(s-\tau),y(s),y(s-\tau),\tilde{\alpha}(s),\overline{\alpha},s)\end{pmatrix}\equiv \tag{14}$$

$$\equiv\begin{pmatrix}f(s)\\ K(t,s)\end{pmatrix}\equiv\tilde{K}(t,s),\quad \psi(\tilde{\alpha}(t),\overline{\alpha},t)=\begin{pmatrix}\psi_x(\tilde{\alpha}(t),\overline{\alpha},t)\\ \psi_y(\tilde{\alpha}(t),\overline{\alpha},t)\end{pmatrix},$$

and obtain IE

$$\tilde{y}(t)=\tilde{r}(\tilde{\alpha}(t),\overline{\alpha},t_0,t)+\int_{t_0}^{t}\tilde{K}(t,\tilde{y}(s),\tilde{y}(s-\tau),\tilde{\alpha}(s),\overline{\alpha},s)ds,\quad t_0\le t\le t^1, \tag{15}$$

$$\tilde{y}(t)=\psi(\tilde{\alpha}(t),\overline{\alpha},t),\ t\in[t_0-\tau,t_0).$$

In further also a notation

$$\eta(t)\equiv\eta(\tilde{y}(t),\tilde{\alpha}(t),\overline{\alpha},t) \tag{16}$$

is used for a model of a measuring device.

6.2. SF with use of an integral model (15)

We generalize results of paper [9] on more general models (15), (16) with variables and constant parameters $\tilde{\alpha}(t),\overline{\alpha}$ and we receive the conjugate equations for basic Lagrange's multipliers $\gamma(t)$

$$\gamma^T(t)=\Phi(t^1)\frac{\partial\tilde{K}(t^1,t)}{\partial\tilde{y}(t)}+\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial\tilde{y}(t)}+\int_{t}^{t^1}\gamma^T(s)\frac{\partial\tilde{K}(s,t)}{\partial\tilde{y}(t)}ds+$$

$$+1(t^1-\tau-t)[\Phi(t^1)\frac{\partial\tilde{K}(t^1,t+\tau)}{\partial\tilde{y}(t)}+\int_{t+\tau}^{t^1}\gamma^T(s)\frac{\partial\tilde{K}(s,t+\tau)}{\partial\tilde{y}(t)}ds],\quad t_0\le t\le t^1. \tag{17}$$

The equation of account of Lagrange's multipliers $\bar{\gamma}(t)$ appropriate to initial function of integral equations with delay time (15)

$$\bar{\gamma}^T(t) = 1(t^1 - \tau - t)[\Phi(t^1)\frac{\partial \widetilde{K}(t^1, t+\tau)}{\partial \widetilde{y}(t)} + \int_{t+\tau}^{t^1} \gamma^T(s)\frac{\partial \widetilde{K}(s, t+\tau)}{\partial \widetilde{y}(t)} ds], \quad t_0 - \tau \le t \le t_0, \qquad (18)$$

are obtained. Here: $\gamma(t)$, $\bar{\gamma}(t)$ are column vectors; $\frac{\partial I_1(t^1)}{\partial \eta(t^1)}\frac{\partial \eta(t^1)}{\partial \widetilde{y}(t^1)} \equiv \Phi(t^1)$; $1(z)$ is single function: it is equal to zero under negative values of argument and is equal to unit under positive values z. These conjugate equations are decided in the opposite direction of time (from t^1).

The form of the conjugate equations doesn't change, but they contain variables and constant parameters $\widetilde{\alpha}(t), \bar{\alpha}$.

From the conjugate equations (17), (18) it is possible to remove single function and to add them a customary aspect.

If $t_0 \le t^1 - \tau \le t^1$, i.e. length of an interval $[t_0, t^1]$ transcends magnitude of a delay time τ, then:

$$\gamma^T(t) = \Phi(t^1)\frac{\partial \widetilde{K}(t^1, t)}{\partial \widetilde{y}(t)} + \frac{\partial f_0(t)}{\partial \eta(t)}\frac{\partial \eta(t)}{\partial \widetilde{y}(t)} + \int_t^{t^1} \gamma^T(s)\frac{\partial \widetilde{K}(s, t)}{\partial \widetilde{y}(t)} ds \quad \text{for } t^1 - \tau \le t \le t^1,$$

$$\gamma^T(t) = \Phi(t^1)\frac{\partial \widetilde{K}(t^1, t)}{\partial \widetilde{y}(t)} + \frac{\partial f_0(t)}{\partial \eta(t)}\frac{\partial \eta(t)}{\partial \widetilde{y}(t)} + \int_t^{t^1} \gamma^T(s)\frac{\partial \widetilde{K}(s, t)}{\partial \widetilde{y}(t)} ds +$$

$$+\Phi(t^1)\frac{\partial \widetilde{K}(t^1, t+\tau)}{\partial \widetilde{y}(t)} + \int_{t+\tau}^{t^1} \gamma^T(s)\frac{\partial \widetilde{K}(s, t+\tau)}{\partial \widetilde{y}(t)} ds \quad \text{for } t_0 \le t \le t^1 - \tau,$$

$$\bar{\gamma}^T(t) = \Phi(t^1)\frac{\partial \widetilde{K}(t^1, t+\tau)}{\partial \widetilde{y}(t)} + \int_{t+\tau}^{t^1} \gamma^T(s)\frac{\partial \widetilde{K}(s, t+\tau)}{\partial \widetilde{y}(t)} ds \quad \text{for } t_0 - \tau \le t \le t_0.$$

If $t^1 - \tau \le t_0$, i.e. the magnitude of delay τ transcends length of an interval $[t_0, t^1]$, (in this case magnitude $t_0 + \tau$ exceeds t^1 – goes out for an interval of object work):

$$\gamma^T(t) = \Phi(t^1)\frac{\partial \widetilde{K}(t^1, t)}{\partial \widetilde{y}(t)} + \frac{\partial f_0(t)}{\partial \eta(t)}\frac{\partial \eta(t)}{\partial \widetilde{y}(t)} + \int_t^{t^1} \gamma^T(s)\frac{\partial \widetilde{K}(s, t)}{\partial \widetilde{y}(t)} ds \quad \text{for } t_0 \le t \le t^1,$$

$$\bar{\gamma}^T(t) = 0 \text{ for } t^1 - \tau \le t \le t_0,$$

$$\bar{\gamma}^T(t) = \Phi(t^1)\frac{\partial \widetilde{K}(t^1, t+\tau)}{\partial \widetilde{y}(t)} + \int_{t+\tau}^{t^1} \gamma^T(s)\frac{\partial \widetilde{K}(s, t+\tau)}{\partial \widetilde{y}(t)} ds \text{ for } t_0 - \tau \le t \le t^1 - \tau.$$

SF for an integral model (15) has the form:

$$\delta I = \delta_{\widetilde{\alpha}(t)} I + \delta_{\widetilde{\alpha}(t^1)} I + \delta_{\overline{\alpha}} I,$$

$$\delta_{\widetilde{\alpha}(t)} I = \int_{t_0}^{t^1} [\frac{\partial f_0(t)}{\partial \eta(t)} \frac{\partial \eta(t)}{\partial \widetilde{\alpha}(t)} + \frac{\partial f_0(t)}{\partial \widetilde{\alpha}(t)} + \gamma^T(t) \frac{\partial \widetilde{r}(t)}{\partial \widetilde{\alpha}(t)} + \Phi(t^1) \frac{\partial \widetilde{K}(t^1, t)}{\partial \widetilde{\alpha}(t)} +$$

$$+ \int_t^{t^1} \gamma^T(s) \frac{\partial \widetilde{K}(s,t)}{\partial \widetilde{\alpha}(t)} ds] \delta \widetilde{\alpha}(t) dt + \int_{t_0 - \tau}^{t_0} \overline{\gamma}^T(t) \frac{\partial \psi(t)}{\partial \widetilde{\alpha}(t)} \delta \widetilde{\alpha}(t) dt;$$

$$\delta_{\widetilde{\alpha}(t^1)} I = [\frac{\partial I_1(t^1)}{\partial \eta(t^1)} \frac{\partial \eta(t^1)}{\partial \widetilde{\alpha}(t^1)} + \Phi(t^1) \frac{\partial \widetilde{r}(t^1)}{\partial \widetilde{\alpha}(t^1)}] \delta \widetilde{\alpha}(t^1);$$

$$\delta_{\overline{\alpha}} I = \left\{ \frac{\partial I_1(t^1)}{\partial \eta(t^1)} \frac{\partial \eta(t^1)}{\partial \overline{\alpha}} + \frac{\partial I_1(t^1)}{\partial \overline{\alpha}} + \Phi(t^1) \left[\frac{\partial \widetilde{r}(t^1)}{\partial \overline{\alpha}} + \int_{t_0}^{t^1} \frac{\partial \widetilde{K}(t^1, s)}{\partial \overline{\alpha}} ds \right] + \right.$$

$$+ \int_{t_0}^{t^1} [\frac{\partial f_0(t)}{\partial \eta(t)} \frac{\partial \eta(t)}{\partial \overline{\alpha}} + \frac{\partial f_0(t)}{\partial \overline{\alpha}} + \gamma^T(t) \frac{\partial \widetilde{r}(t)}{\partial \overline{\alpha}} + \int_t^{t^1} \gamma^T(s) \frac{\partial \widetilde{K}(s,t)}{\partial \overline{\alpha}} ds] dt + \int_{t_0 - \tau}^{t_0} \overline{\gamma}^T(t) \frac{\partial \psi(t)}{\partial \overline{\alpha}} dt +$$

$$+ \left[\Phi(t^1) [\frac{\partial \widetilde{r}(t^1)}{\partial t_0} - \widetilde{K}(t^1, t_0) + 1(t^1 - t_0 - \tau)(\widetilde{K}(t^1, t_0 + \tau - 0) - \widetilde{K}(t^1, t_0 + \tau + 0))] + \right.$$

$$- f_0(t_0) + \int_{t_0}^{t^1} \gamma^T(t) (\frac{\partial \widetilde{r}(t)}{\partial t_0} - \widetilde{K}(t, t_0)) dt +$$

$$\left. + \ 1(t^1 - t_0 - \tau) \int_{t_0 + \tau}^{t^1} \gamma^T(t) [\widetilde{K}(t, t_0 + \tau - 0) - \widetilde{K}(t, t_0 + \tau + 0)] dt \right] \frac{dt_0}{d\overline{\alpha}} +$$

$$+ \left[\Phi(t^1) [\frac{\partial \widetilde{r}(t^1)}{\partial t^1} + \widetilde{K}(t^1, t^1) + \int_{t_0}^{t^1} \frac{\partial \widetilde{K}(t^1, s)}{\partial t^1} ds] + \right. \tag{19}$$

$$\left. + \frac{\partial I_1(t^1)}{\partial \eta(t^1)} \frac{\partial \eta(t^1)}{\partial t^1} + \frac{\partial I_1(t^1)}{\partial t^1} + f_0(t^1) \right] \frac{dt^1}{d\overline{\alpha}} ++$$

$$\left[\Phi(t^1) [1(t^1 - t_0 - \tau)(\widetilde{K}(t^1, t_0 + \tau - 0) - \widetilde{K}(t^1, t_0 + \tau + 0)) - \right.$$

$$- \int_{t_0}^{t^1} \frac{\partial \widetilde{K}(t^1, s)}{\partial \widetilde{y}(s - \tau)} \frac{d \widetilde{y}(s - \tau)}{d(s - \tau)} ds] +$$

$$+ 1(t^1 - t_0 - \tau) \int_{t_0 + \tau}^{t^1} \gamma^T(t) [\widetilde{K}(t, t_0 + \tau - 0) - \widetilde{K}(t, t_0 + \tau + 0)] dt -$$

$$-\int_{t_0}^{t^1}\gamma^T(t)\int_{t_0}^{t}\frac{\partial\tilde{K}(t,s)}{\partial\tilde{y}(s-\tau)}\frac{d\tilde{y}(s-\tau)}{d(s-\tau)}ds\,dt\Bigg]\frac{d\tau}{d\overline{\alpha}}\Bigg\}d\overline{\alpha}.$$

6.3. SF with use of an integro-differential model (1)

It is necessary in (17)–(19) to fulfil matrix transformations (differentiation, multiplication) with the registration earlier entered notations (14), and also

$$\gamma(t)=\begin{pmatrix}\gamma_x(t)\\ \gamma_y(t)\end{pmatrix},\ \gamma^T(t)=\left(\gamma_x^T(t);\ \gamma_y^T(t)\right),\ \overline{\gamma}(t)=\begin{pmatrix}\overline{\gamma}_x(t)\\ \overline{\gamma}_y(t)\end{pmatrix},\ \overline{\gamma}^T(t)=\left(\overline{\gamma}_x^T(t);\ \overline{\gamma}_y^T(t)\right),$$

i.e. $$\Phi(t^1)\equiv\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial\tilde{y}(t^1)}=\left(\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial x(t^1)};\ \frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial y(t^1)}\right)\equiv\left(\Phi_x(t^1);\ \Phi_y(t^1)\right),$$

$$\frac{\partial\tilde{K}(t^1,t)}{\partial\tilde{y}(t)}=\begin{pmatrix}\dfrac{\partial f(t)}{\partial x(t)}; & \dfrac{\partial f(t)}{\partial y(t)}\\[2ex] \dfrac{\partial K(t^1,t)}{\partial x(t)}; & \dfrac{\partial K(t^1,t)}{\partial y(t)}\end{pmatrix},\ \frac{\partial\tilde{r}(t^1)}{\partial\overline{\alpha}}=\begin{pmatrix}\dfrac{\partial x_0(\overline{\alpha},t_0)}{\partial\overline{\alpha}}\\[2ex] \dfrac{\partial r(t^1)}{\partial\overline{\alpha}}\end{pmatrix},$$

$$\Phi(t^1)\frac{\partial\tilde{K}(t^1,t)}{\partial\tilde{y}(t)}=\left(\Phi_x(t^1)\frac{\partial f(t)}{\partial x(t)}+\Phi_y(t^1)\frac{\partial K(t^1,t)}{\partial x(t)};\ \Phi_x(t^1)\frac{\partial f(t)}{\partial y(t)}+\Phi_y(t^1)\frac{\partial K(t^1,t)}{\partial y(t)}\right),$$

$$\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial\tilde{y}(t)}=\left(\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial x(t)};\ \frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial y(t)}\right),$$

$$\gamma^T(s)\frac{\partial\tilde{K}(s,t)}{\partial\tilde{y}(t)}=\left(\gamma_x^T(s)\frac{\partial f(t)}{\partial x(t)}+\gamma_y^T(s)\frac{\partial K(s,t)}{\partial x(t)};\ \gamma_x^T(s)\frac{\partial f(t)}{\partial y(t)}+\gamma_y^T(s)\frac{\partial K(s,t)}{\partial y(t)}\right),$$

$$\Phi(t^1)\frac{\partial\tilde{r}(t^1)}{\partial\overline{\alpha}}=\Phi_x(t^1)\frac{\partial x_0(\overline{\alpha},t_0)}{\partial\overline{\alpha}}+\Phi_y(t^1)\frac{\partial r(t^1)}{\partial\overline{\alpha}}, \tag{20}$$

$$\gamma^T(t)\frac{\partial\tilde{r}(t)}{\partial\overline{\alpha}}=\gamma_x^T(t)\frac{\partial x_0(\overline{\alpha},t_0)}{\partial\overline{\alpha}}+\gamma_y^T(t)\frac{\partial r(t)}{\partial\overline{\alpha}},$$

$$\Phi(t^1)\frac{\partial\tilde{K}(t^1,t)}{\partial\overline{\alpha}}=\Phi_x(t^1)\frac{\partial f(t)}{\partial\overline{\alpha}}+\Phi_y(t^1)\frac{\partial K(t^1,t)}{\partial\overline{\alpha}},$$

$$\int_t^{t^1}\gamma^T(s)\frac{\partial\tilde{K}(s,t)}{\partial\overline{\alpha}}ds=\int_t^{t^1}[\gamma_x^T(s)\frac{\partial f(t)}{\partial\overline{\alpha}}+\gamma_y^T(s)\frac{\partial K(s,t)}{\partial\overline{\alpha}}]ds,\ \text{etc.}$$

In the total we obtain the conjugate equations for Lagrange's multipliers

$$\gamma_x^T(t) = \Phi_x(t^1)\frac{\partial f(t)}{\partial x(t)} + \Phi_y(t^1)\frac{\partial K(t^1,t)}{\partial x(t)} + \frac{\partial f_0(t)}{\partial \eta(t)}\frac{\partial \eta(t)}{\partial x(t)} +$$

$$+\int_t^{t^1}[\gamma_x^T(s)\frac{\partial f(t)}{\partial x(t)} + \gamma_y^T(s)\frac{\partial K(s,t)}{\partial x(t)}]ds +$$

$$+1(t^1 - \tau - t)[\Phi_x(t^1)\frac{\partial f(t+\tau)}{\partial x(t)} + \Phi_y(t^1)\frac{\partial K(t^1,t+\tau)}{\partial x(t)} +$$

$$+\int_{t+\tau}^{t^1}[\gamma_x^T(s)\frac{\partial f(t+\tau)}{\partial x(t)} + \gamma_y^T(s)\frac{\partial K(s,t+\tau)}{\partial x(t)}]ds\ ],$$

$$\gamma_y^T(t) = \Phi_x(t^1)\frac{\partial f(t)}{\partial y(t)} + \Phi_y(t^1)\frac{\partial K(t^1,t)}{\partial y(t)} + \frac{\partial f_0(t)}{\partial \eta(t)}\frac{\partial \eta(t)}{\partial y(t)} + \tag{21}$$

$$+\int_t^{t^1}[\gamma_x^T(s)\frac{\partial f(t)}{\partial y(t)} + \gamma_y^T(s)\frac{\partial K(s,t)}{\partial y(t)}]ds +$$

$$+1(t^1 - \tau - t)[\Phi_x(t^1)\frac{\partial f(t+\tau)}{\partial y(t)} + \Phi_y(t^1)\frac{\partial K(t^1,t+\tau)}{\partial y(t)} +$$

$$+\int_{t+\tau}^{t^1}[\gamma_x^T(s)\frac{\partial f(t+\tau)}{\partial y} + \gamma_y^T(s)\frac{\partial K(s,t+\tau)}{\partial y}]ds\ ],\quad t_0 \le t \le t^1;$$

$$\bar{\gamma}_x^T(t) = 1(t^1 - \tau - t)[\Phi_x(t^1)\frac{\partial f(t+\tau)}{\partial x(t)} + \Phi_y(t^1)\frac{\partial K(t^1,t+\tau)}{\partial x(t)} +$$

$$+\int_{t+\tau}^{t^1}[\gamma_x^T(s)\frac{\partial f(t+\tau)}{\partial x(t)} + \gamma_y^T(s)\frac{\partial K(s,t+\tau)}{\partial x(t)}]ds\ ],$$

$$\bar{\gamma}_y^T(t) = 1(t^1 - \tau - t)[\Phi_x(t^1)\frac{\partial f(t+\tau)}{\partial y(t)} + \Phi_y(t^1)\frac{\partial K(t^1,t+\tau)}{\partial y(t)} +$$

$$+\int_{t+\tau}^{t^1}[\gamma_x^T(s)\frac{\partial f(t+\tau)}{\partial y(t)} + \gamma_y^T(s)\frac{\partial K(s,t+\tau)}{\partial y(t)}]ds\ ],\ t_0 - \tau \le t \le t_0.$$

The first variation of a functionality I in relation to variable $\tilde{\alpha}(t)$ and constant $\tilde{\alpha}(t^1)$, $\bar{\alpha}$ parameters, has three components - watch a formula (5):

$$\delta I = \delta_{\tilde{\alpha}(t)} I + \delta_{\tilde{\alpha}(t^1)} I + \delta_{\bar{\alpha}} I; \tag{22}$$

$$\delta_{\tilde{\alpha}(t)} I = \int_{t_0}^{t^1} \left[\frac{\partial f_0(t)}{\partial \eta(t)} \frac{\partial \eta(t)}{\partial \tilde{\alpha}(t)} + \frac{\partial f_0(t)}{\partial \tilde{\alpha}(t)} + \gamma_y^T(t) \frac{\partial r(t)}{\partial \tilde{\alpha}(t)} + \Phi_x(t^1) \frac{\partial f(t)}{\partial \tilde{\alpha}(t)} + \Phi_y(t^1) \frac{\partial K(t^1,t)}{\partial \tilde{\alpha}(t)} + \right.$$

$$\left. + \int_t^{t^1} [\gamma_x^T(s) \frac{\partial f(t)}{\partial \tilde{\alpha}(t)} + \gamma_y^T(s) \frac{\partial K(s,t)}{\partial \tilde{\alpha}(t)}] ds \right] \delta\tilde{\alpha}(t)\, dt +$$

$$+ \int_{t_0-\tau}^{t_0} [\bar{\gamma}_x^T(t) \frac{\partial \psi_x(t)}{\partial \tilde{\alpha}(t)} + \bar{\gamma}_y^T(t) \frac{\partial \psi_y(t)}{\partial \tilde{\alpha}(t)}] \delta\tilde{\alpha}(t)\, dt;$$

$$\delta_{\tilde{\alpha}(t^1)} I = [\frac{\partial I_1(t^1)}{\partial \eta(t^1)} \frac{\partial \eta(t^1)}{\partial \tilde{\alpha}(t^1)} + \Phi_y(t^1) \frac{\partial r(t^1)}{\partial \tilde{\alpha}(t^1)}] \delta\tilde{\alpha}(t^1);$$

$$\delta_{\bar{\alpha}} I = \left\{ \frac{\partial I_1(t^1)}{\partial \eta(t^1)} \frac{\partial \eta(t^1)}{\partial \bar{\alpha}} + \frac{\partial I_1(t^1)}{\partial \bar{\alpha}} + \right.$$

$$+ \Phi_x(t^1) \left[\frac{\partial x_0(\bar{\alpha}, t_0)}{\partial \bar{\alpha}} + \int_{t_0}^{t^1} \frac{\partial f(s)}{\partial \bar{\alpha}} ds \right] + \Phi_y(t^1) \left[\frac{\partial r(t^1)}{\partial \bar{\alpha}} + \int_{t_0}^{t^1} \frac{\partial K(t^1,s)}{\partial \bar{\alpha}} ds \right] +$$

$$+ \int_{t_0}^{t^1} \left[\frac{\partial f_0(t)}{\partial \eta(t)} \frac{\partial \eta(t)}{\partial \bar{\alpha}} + \frac{\partial f_0(t)}{\partial \bar{\alpha}} + \gamma_x^T(t) \frac{\partial x_0(\bar{\alpha}, t_0)}{\partial \bar{\alpha}} + \gamma_y^T(t) \frac{\partial r(t)}{\partial \bar{\alpha}} + \right.$$

$$\left. + \int_t^{t^1} [\gamma_x^T(s) \frac{\partial f(t)}{\partial \bar{\alpha}} + \gamma_y^T(s) \frac{\partial K(s,t)}{\partial \bar{\alpha}}] ds \right] dt + \int_{t_0-\tau}^{t_0} [\bar{\gamma}_x^T(t) \frac{\partial \psi_x(t)}{\partial \bar{\alpha}} + \bar{\gamma}_y^T(t) \frac{\partial \psi_y(t)}{\partial \bar{\alpha}}] dt +$$

$$+ \left[\Phi_x(t^1) [\frac{\partial x_0(\bar{\alpha}, t_0)}{\partial t_0} - f(t_0) + 1(t^1 - t_0 - \tau)(f(t_0 + \tau - 0) - f(t_0 + \tau + 0))] + \right.$$

$$+ \Phi_y(t^1) [\frac{\partial r(t^1)}{\partial t_0} - K(t^1, t_0) + 1(t^1 - t_0 - \tau)(K(t^1, t_0 + \tau - 0) - K(t^1, t_0 + \tau + 0))] - f_0(t_0) +$$

$$+ \int_{t_0}^{t^1} \gamma_x^T(t)\, dt (\frac{\partial x_0(\bar{\alpha}, t_0)}{\partial t_0} - f(t_0)) + \int_{t_0}^{t^1} \gamma_y^T(t) (\frac{\partial r(t)}{\partial t_0} - K(t, t_0))\, dt +$$

$$+ 1(t^1 - t_0 - \tau) \int_{t_0+\tau}^{t^1} \gamma_x^T(t)\, dt [f(t_0 + \tau - 0) - f(t_0 + \tau + 0)] +$$

$$\left. + \ 1(t^1 - t_0 - \tau) \int_{t_0+\tau}^{t^1} \gamma_y^T(t) [K(t, t_0 + \tau - 0) - K(t, t_0 + \tau + 0)]\, dt \right] \frac{dt_0}{d\bar{\alpha}} +$$

$$+\Bigg[\Phi_x(t^1)f(t^1)+\Phi_y(t^1)[\frac{\partial r(t^1)}{\partial t^1}+K(t^1,t^1)+\int_{t_0}^{t^1}\frac{\partial K(t^1,s)}{\partial t^1}ds]+$$

$$+\frac{\partial I_1(t^1)}{\partial \eta(t^1)}\frac{\partial \eta(t^1)}{\partial t^1}+\frac{\partial I_1(t^1)}{\partial t^1}+f_0(t^1)\Bigg]\frac{dt^1}{d\overline{\alpha}}+$$

$$+\Big[\Phi_x(t^1)[1(t^1-t_0-\tau)(f(t_0+\tau-0)-f(t_0+\tau+0))-$$

$$-\int_{t_0}^{t^1}(\frac{\partial f(s)}{\partial x(s-\tau)}\frac{dx(s-\tau)}{d(s-\tau)}+\frac{\partial f(s)}{\partial y(s-\tau)}\frac{dy(s-\tau)}{d(s-\tau)})ds]+$$

$$+\Phi_y(t^1)[1(t^1-t_0-\tau)(K(t^1,t_0+\tau-0)-K(t^1,t_0+\tau+0))-$$

$$-\int_{t_0}^{t^1}(\frac{\partial K(t^1,s)}{\partial x(s-\tau)}\frac{dx(s-\tau)}{d(s-\tau)}+\frac{\partial K(t^1,s)}{\partial y(s-\tau)}\frac{dy(s-\tau)}{d(s-\tau)})ds]+$$

$$+1(t^1-t_0-\tau)\int_{t_0+\tau}^{t^1}\gamma_x^T(t)dt[f(t_0+\tau-0)-f(t_0+\tau+0)]+$$

$$+1(t^1-t_0-\tau)\int_{t_0+\tau}^{t^1}\gamma_y^T(t)[K(t,t_0+\tau-0)-K(t,t_0+\tau+0)]dt-$$

$$-\int_{t_0}^{t^1}\gamma_x^T(t)\int_{t_0}^{t}\left(\frac{\partial f(s)}{\partial x(s-\tau)}\frac{dx(s-\tau)}{d(s-\tau)}+\frac{\partial f(s)}{\partial y(s-\tau)}\frac{dy(s-\tau)}{d(s-\tau)}\right)ds\,dt-$$

$$-\int_{t_0}^{t^1}\gamma_y^T(t)\int_{t_0}^{t}\left(\frac{\partial K(t,s)}{\partial x(s-\tau)}\frac{dx(s-\tau)}{d(s-\tau)}+\frac{\partial K(t,s)}{\partial y(s-\tau)}\frac{dy(s-\tau)}{d(s-\tau)}\right)ds\,dt\Bigg]\frac{d\tau}{d\overline{\alpha}}\Bigg\}d\overline{\alpha}.$$

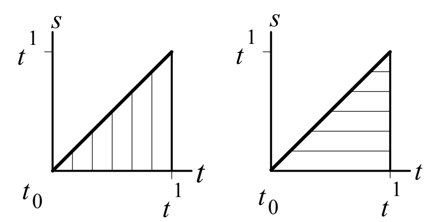

Fig. 7. Triangular area and order of an integration

It is expedient to add the conjugate equations for Lagrange's multipliers (21) too form of the integro-differential equations.

We enter new variable $\Phi_x(t^1)+\int_t^{t^1}\gamma_x^T(s)ds=\lambda_x^T(t)$, either

$\gamma_x^T(t) = -\dot{\lambda}_x^T(t)$, $\lambda_x^T(t^1) = \Phi_x(t^1)$, and change an order of integrating in double integral inside of triangular area (see fig. 7) $\left(\text{i.e.} \int_{t_0}^{t^1}\int_{t_0}^{t} A(t,s)\,ds\,dt = \int_{t_0}^{t^1}\int_{t}^{t^1} A(s,t)\,ds\,dt \right)$. Then conjugate equations (21) are noted as (4) and SF (see (22)) are calculated under the formula (5).

Discussion

In this paper the variational method of calculation SF and SC for the multivariate nonlinear dynamic systems described by general continuous vectorial Volterra's integro-differential equations of the second genus with dead time is developed.

The variational method is based on invariant expansion of initial functional for system due to inclusion in it of the dynamic equations of object model and of measuring device model with the help of Lagrange's multipliers and on computation of the first variation expanded functional on phase coordinates of model and in required parameters. The equating with a zero of the functions facing to variations of phase coordinates gives the dynamic equations for Lagrange's multipliers. The simplified first variation represents required sensitivity functional.

Novelty and generality of results consists in a generality of dynamic object model , of the measuring device model and of quality functional. In models both variables and constant parameters are present. In a basis of calculation of sensitivity indexes the decision of the integro-differential equations of object model in a forward direction of time and obtained integro-differential equations for Lagrange's multipliers in the opposite direction of time lays.

Received in paper SF are more general in comparison with known in the scientific literature.

It is shown, that by consideration of more simple dynamic systems it is enough to put in the received results to zero corresponding additives – look sections 4 and 5.

Results are applicable at the decision of problems of identification, adaptive optimal control and optimization of dynamic systems, i.e. they allow to create precision systems and devices.

Conclusion

Variational method of calculation of SC and SF allows to do a generalization on more complex dynamic system classes, described by: integral and integro-differential equations with additional some dead times and different classes of discontinuous dynamic equations.

References

1. G.M. Ostrovskij, Yu.M. Volin. *Methods of optimization of chemical reactors*, "Chemistry" Publishing House, Moscow, 1967.
2. C.B. Speedy, R.F. Brown, G. C. Goodwin. *Control theory: identification and optimal control*, "Mir" Publishing House, Moscow, 1973. (Oliverand Boyd, Edinburg, 1970)
3. *Methods of sensitivity theory in automatic control* / Under editing *E.N. Rosenvasser* and *R.M. Yusupov*, "Energy" Publishing House, Leningrad, 1971.
4. A.E. Bryson, Ju-Chi Ho. *Applied optimal control,* "Mir" Publishing House, Moscow, 1972. (Blaisdell Publishing Company; Waltham, Massachusetts, Toronto, London, 1969).
5. A.I. Ruban. *Nonlinear dynamic object identification on the base of sensitivity algorithm*, Tomsk University Publisher, Tomsk, 1975.
6. E.N. Rosenvasser, R.M. Yusupov. *Sensitivity of control systems,* "Nauka" Publishing House, Moscow, 1981.
7. A.I. Ruban. *Identification and sensitivity of complex systems,* Tomsk University Publisher, Tomsk, 1982.
8. A.I. Rouban, "Coefficients and functionals of sensitivity for multivariate systems described by integral and integro-differetial equations", AMSE Journals, Series Advances A, vol. 35, no. 1, pp. 25–34, 1999.
9. A.I. Rouban. "Coefficients and functional of sensitivity for dynamic systems described by integral equations with dead time", AMSE Journals, Series Advances C, vol. 57, no. 3, pp. 15–34. 2002.
10. R. Bellman, K.L. Cooke, *Differential-difference equations*, "Mir" Publishing House, Moscow, 1967. (New York, London; Academic Press, 1963)
11. A.I. Rouban. "Sensitivity coefficients for many-dimensional continuous and discontinuous dynamic systems with delay time", AMSE Journals, Series Advances A, vol. 36, no. 2, pp. 17–36. 1999.
12. A.I. Rouban "Sensitivity functionals in the Bolts's problem for the maltivariate dynamic systems described by the integro-differential equations with delay", *Control Sciences*, no. 2, pp. 2–8, 2013.